The Cauchy Problem in Kinetic Theory

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Preface

This book stems from lecture notes for a course in Kinetic Theory I gave at Indiana University in the spring of 1993. The class was composed of several of my colleagues from the faculty and advanced graduate students, most of whom were writing theses in partial differential equations. My goal was to introduce them to the study of the Cauchy problem for the Boltzmann and Vlasov equations.

The desired results on both equations are scattered throughout the literature. Thus one of the purposes of this book is to collect such results in one place. When these notes first appeared (May 1993) there were no similar texts available. Since that time, the excellent reference [4] on the Boltzmann equation has been published. One finds in the literature that the starting point for many studies (e.g., solutions near the equilibrium) assumes the reader is familiar with a large amount of background material. For instance, the work of Grad [13] on the linearized Boltzmann problem is a "given" in many papers. We have tried to fill in these gaps and unify the presentation.

We claim no attempt at complete generality here. Thus, for instance, when studying the Boltzmann problem, we treat the "hard-sphere" case whenever convenient. In the chapter on small-data solvability for the Vlasov–Maxwell system, we assume that each of the plasma densities is initially small, although there is a significant generalization [9] to the "nearly neutral" case in which cancellation is taken into account.

We begin in Chapter 1 with a study of the collision operator and "crude" derivations of the equations to be considered. All such equations start with the Liouville equation. Related material on conservation laws and the entropy is also presented in this introductory chapter.

Chapter 2 deals with solvability near the vacuum for the Boltzmann equation. It is an adaptation of the works of Illner and Shinbrot [15] and Polewczak [21]. This concerns the hard-sphere case; global solutions are obtained for "small" Cauchy data (near 0).

In the following chapter we study the solvability of the Boltzmann equation for Cauchy data near the equilibrium. In order to unify the treatment, we include here Grad's computation [13] of the integral operator for the linearized equation. One begins by linearizing about a homogeneous equilibrium, a Maxwellian distribution of velocities. Then we follow the recent paper of Kawashima [16]; this is self-contained and elegant. I felt it would be less time consuming to develop Kawashima's ideas than to refer to the original works of Ukai [23], [24], [25], Nishida and Imai [19] and Ellis and Pinsky [7]. It is shown that Cauchy data sufficiently near a Maxwellian launches a global solution to the full nonlinear Boltzmann equation.

In the following chapters we switch to the study of collisionless plasmas and Vlasov problems. In Chapter 4 we consider the Vlasov–Poisson system. This has been recently solved globally in time for large data by Pfaffelmoser [20]. Simpler proofs have since been given by Schaeffer [22], Horst [14], Lions and Perthame [18] and Wollman [26]. An unpublished variation of the proof of Schaeffer [22] is presented, which was kindly communicated to me by Schaeffer.

We then turn in Chapter 5 to the Vlasov–Maxwell system. Here, as for the Vlasov–Poisson equation, a sufficient condition for global classical solvability is known: that one be able to control the influence of large velocities. This appears in [11]. However, at this point such a bound has been constructed only for "small" or "nearly neutral" data. Global solutions to the small-data Cauchy problem are then obtained in Chapter 6. This material is taken from [12].

The next chapter concerns the smoothing property of velocity averages and global weak solutions to the Vlasov–Maxwell system. This is based on work of DiPerna and Lions [6] as modified by Kruse [17].

In the last chapter we present a particle method for the numerical approximation of the "one and one–half dimensional" relativistic Vlasov–Maxwell system. Here the phase–space density f is a function of the time t, one space variable x and two velocity variables v_1, v_2; this is the "smallest" system for which one has a nontrivial magnetic field. We use a special algorithm from [10]; it is noteworthy that finite differences are *not* used to advance the fields in time. We obtain essentially first–order convergence.

There are many omissions. In particular, we omit the study of "soft" potentials in the near equilibrium solution of the Boltzmann problem (see e.g., [2]). The initial–value problem for the spatially homogeneous Boltzmann problem is not considered. For this and related perturbative results for "weakly inhomogeneous" problems, see [1] and [4]. The Enskog equation is not discussed at all, nor are boundary–value problems of any type. The elegant proof of global existence and regularity for the Vlasov–Poisson system, due to Lions and Perthame [18], is also omitted. Many advances have recently been made in the study of stability of stationary solutions for Vlasov problems; we refer the reader to the references in later chapters. Also, we do not include a treatment of weak solvability of the Boltzmann problem. This can be found in [5], [8] and [4]. However, the essential compactness used there is contained in our Chapter 7.

Surely a project of this magnitude is likely to contain errors. I plan to post a list of errata which may be obtained via anonymous ftp at

`iu-math.math.indiana.edu`

in the directory **pub/glassey**. One might also check the URL

http://www.math.indiana.edu/

in the future for further information.

I wish to thank my class for a stimulating experience. Its members included Mark Daniel, Xiaoqiu Gao, Sang–woo Heo, Chi Shun Kwong, Raymond Lai, Gheorge Minea, Hongjie Ni, Ricardo Rosa, Xiaoming Wang, Kang Xue and Mei–Qin Zhan. I am also grateful for comments from my colleagues David Hoff, Mohammed Khodja, Shouhong Wang and Kevin Zumbrun.

I wish to specially acknowledge the contributions of my friends and coauthors Walter Strauss of Brown University and Jack Schaeffer of Carnegie Mellon University. I am indebted to H. Andréasson, R. Illner and B. Perthame for constructive comments and to J. Batt for introducing me to Vlasov problems in 1975. I thank Ms. Vicki Botos and Ms. Mary Jane Wilcox for expertly rendering parts of a (poorly) handwritten manuscript into TeX form. Finally my family deserves recognition for their patience and understanding in allowing me to complete this project.

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Chapter 1

Properties of the Collision Operator

1.1. Kinetic Theory, Derivation of the Equations

By "kinetic theory" we understand a mathematical model in which a gas is represented as a collection of molecules whose motion in "phase space" is to be analyzed. Phase space is the Cartesian product of three dimensional position space with three dimensional velocity space. We use a statistical approach and posit the existence of a "velocity distribution function" $f = f(t, x, v)$ where $t > 0$; $x, v \in \mathbb{R}^3$. Here $f \geq 0$, $f \to 0$, $|v| \to \infty$ and the probable number of molecules which, at time t, are situated in a volume element $x, x+dx$ having velocities in v, $v+dv$, is $f(t, x, v)\, dv\, dx$. The distribution function f contains an immense amount of information, so one can then use f to calculate macroscopic properties. Standard references on kinetic theory include [9], [11], [16], [17], [22], and [31]. For other mathematical treatments of the Boltzmann equation, see [10] and [32].

The areas we will study include:

1. **Rarefied Gases** (Boltzmann equation (1872), [7])

The assumptions are:

(i) the gas is electrically neutral;
(ii) the mean distance between molecules is large in comparison to their size; i.e., in comparison to the range of intermolecular forces;
(iii) encounters with other molecules form a very small part of the lifetime of a molecule; therefore only binary collisions are important;
(iv) collisions preserve mass, momentum and energy.

2. **Plasmas (completely ionized gases)**

There are essentially two theories: Vlasov (collisionless) and MHD (magnetohydrodynamics). Their properties may be summarized in the following table:

	VLASOV	*MHD*
time scale	rapid	slow
temperature	high	low
density	low	high
collisions	ignored	very important

Table 1.1.*Physical Characteristics of Vlasov vs MHD Plasmas*

Here "slow" means the fluid motion is slow with respect to the thermal motion of molecules; "high temperature" means $T \gg e^2/\bar{r}$ where

$$
\begin{aligned}
-e &= \text{charge of an electron} \\
T &= \text{temperature} \\
\bar{r} &= \text{mean distance between molecules, etc.}
\end{aligned}
$$

As examples we list for MHD fusion reactors and stars, while typical Vlasov plasmas include the Van Allen radiation belts, nebulae, the solar wind and (one of the two) tails of a comet.

All equations stem from the **Liouville equation**:

$$
\frac{\mathcal{D}f}{\mathcal{D}t} = \text{“material derivative”} = \begin{array}{l}\text{rate of change along particle} \\ \text{paths in phase space } \mathbb{R}^3 \times \mathbb{R}^3\end{array} \tag{1.1}
$$

Thus

$$
\frac{\mathcal{D}f}{\mathcal{D}t} = \begin{array}{l}\text{rate of change due} \\ \text{to collisions}\end{array} \equiv C(f). \tag{1.2}
$$

If we call $(\dot{x}, \dot{v})$ the velocity in phase space then by Newton's equations of motion

$$
\begin{aligned}
\dot{x} &= \textbf{velocity} = v \\
\dot{v} &= \textbf{force} = F
\end{aligned} \tag{1.3}
$$

and the Liouville equation can be written as

$$
\partial_t f + \nabla_x f \cdot \dot{x} + \nabla_v f \cdot \dot{v} = C(f)
$$

or

$$
\partial_t f + v \cdot \nabla_x f + F \cdot \nabla_v f = C(f). \tag{1.4}
$$

1.1.1. "Derivation" of the Vlasov–Maxwell System. We ignore collisions: $C(f) = 0$. Plasma dynamics are electromagnetic in nature; hence we couple the Maxwell system to the Liouville equation. Let c denote the speed of light. From the study of electricity and magnetism, we have

$$\begin{aligned} \dot{x} &= v \\ \dot{v} &= \textbf{Lorentz force } = E + \frac{v}{c} \times B. \end{aligned} \tag{1.5}$$

Then the **Vlasov equation** (1946) is

$$\partial_t f + v \cdot \nabla_x f + \Big(E + \frac{v}{c} \times B\Big) \cdot \nabla_v f = 0. \tag{1.6}$$

Maxwell's equations are

$$\begin{aligned} c^{-1} E_t &= \nabla \times B - 4\pi c^{-1} j \qquad & \nabla \cdot E &= 4\pi\rho \\ c^{-1} B_t &= -\nabla \times E & \nabla \cdot B &= 0. \end{aligned}$$

Here j is called the **current density**, ρ is the **charge density**. Both will be specified below. In order to see the coupling, we recall that the constraint on the divergence of E is preserved in time provided that

$$\rho_t + \nabla_x \cdot j = 0 \qquad \begin{pmatrix} \text{conservation} \\ \text{of charge} \end{pmatrix}. \tag{1.7}$$

Indeed, formally we have

$$\begin{aligned} \frac{\partial}{\partial t}(\nabla_x \cdot E - 4\pi\rho) &= \nabla_x \cdot E_t - 4\pi\rho_t \\ &= \nabla_x \cdot (c\nabla_x \times B - 4\pi j) - 4\pi\rho_t \\ &= -4\pi\rho_t - \nabla_x \cdot 4\pi j = 0. \end{aligned}$$

Similarly, the field B remains divergence free if it is so initially. Now integrate the Vlasov equation (1.6) with respect to v: if f vanishes with sufficient rapidity at ∞, we get

$$\partial_t \int f\, dv + \int v \cdot \nabla_x f\, dv = 0 \tag{1.8}$$

because

$$\Big(E + \frac{v}{c} \times B\Big) \cdot \nabla_v f = \nabla_v \cdot \Big[\Big(E + \frac{v}{c} \times B\Big) f\Big]. \tag{1.9}$$

In view of (1.7), this suggests that we take

$$\begin{aligned} \rho(t,x) &= \int f\, dv \\ j(t,x) &= \int v f\, dv. \end{aligned} \tag{1.10}$$

Actually we take 2 species: electrons, with density f_e, and ions, with density f_i with the electron charge equal to $-e$, and the ion charge equal to ze (here $z \in \mathbb{N}$). Then the Vlasov equation takes the form

$$\begin{aligned}
&\partial_t f_e + v \cdot \nabla_x f_e - e\left(E + \frac{1}{c} v \times B\right) \cdot \nabla_v f_e = 0 \\
&\partial_t f_i + v \cdot \nabla_x f_i + ze\left(E + \frac{1}{c} v \times B\right) \cdot \nabla_v f_i = 0
\end{aligned} \tag{1.11}$$

and in the Maxwell equations we take

$$\begin{aligned}
\rho &= e \int_{\mathbb{R}^3} (z f_i - f_e)\, dv \\
j &= e \int_{\mathbb{R}^3} v(z f_i - f_e)\, dv.
\end{aligned} \tag{1.12}$$

1.1.2. A formal "derivation" of the Boltzmann Equation. Let the mass be normalized to unity. Consider a two particle collision, with one particle having values of velocities in a range dv, the other with values of velocities in a range du. In a collision, these acquire values of velocities in the ranges dv', du' respectively.

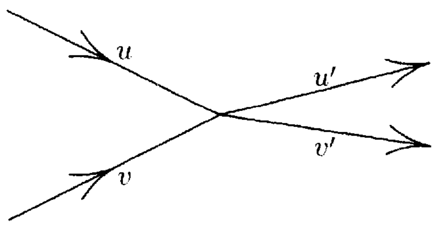

Fig. 1.1 *Collisions*

Collisions conserve *momentum*

$$u' + v' = u + v \tag{1.13}$$

and *energy*

$$|u'|^2 + |v'|^2 = |u|^2 + |v|^2. \tag{1.14}$$

Now the total number of collisions per unit time per unit volume is taken to be

$$\underbrace{\{\text{the number of particles/unit volume}\}}_{f(t,x,v)dv} \times \underbrace{\left\{\begin{matrix}\text{probability any} \\ \text{of them suffers a} \\ \text{collision}\end{matrix}\right\}}_{p}.$$

One takes p proportional to

$$\underbrace{\{\text{the number of particles/unit volume}\}}_{f(t,x,u)du} \times \{du' \times dv'\}.$$

Thus the

$$\frac{\text{total number of collisions}}{(\text{unit volume})(\text{unit time})} = w(u', v'; u, v) f(u) f(v)\, du\, dv\, du'\, dv'.$$

Here w is determined from analytical mechanics by solving the collision problem assuming a given intermolecular force. It is also conventional to abbreviate $f(t, x, u)$ by $f(u)$, etc. From [31] one learns that Maxwell himself assumed that the probability density for a pair of molecules with velocities v, u at (t, x) is proportional to the product $f(t, x, u) f(t, x, v)$. This hypothesis is called "molecular chaos" and is recognized to be that of stochastic independence.

Symmetry for w is achieved via the "Principle of detailed balancing" which asserts that

$$w(u', v'; u, v) = w(u, v; u', v'). \tag{1.15}$$

This is formally discussed in the physics books in the references. Suffice it to say the following. In equilibrium, the number of collisions $(u, v) \mapsto (u', v')$ is equal to the number of collisions $(-u', -v') \mapsto (-u, -v)$. This follows from symmetry of the equations of classical mechanics under time reversal, and is adopted in nonequilibrium settings also. Thus under such a mapping we expect to get

$$w(u', v', u, v) = w(-u, -v; -u', -v') \tag{1.16}$$

and then the stated result.

1.2. The Form of the Collision Operator

Let two molecules collide. Every such collision transfers it out of a particular range dv (*losses*). Given dv, the total number of collisions $(u, v) \mapsto (u', v')$ with all possible values of u, u', v' occurring in the volume dx per unit time is

$$dx\, dv \cdot \int w(u', v'; u, v) f(u) f(v)\, du\, du'\, dv'. \tag{1.17}$$

There are also *gains*: collisions which bring into the range dv molecules which originally had values outside that range. Given v, these are collisions $(u', v') \mapsto (u, v)$ with all possible u, u', v', and

(1.18)

$$\frac{\begin{array}{c}\text{the total number of such}\\ \text{collisions in the volume } dx\end{array}}{(\text{unit time})} = dx\, dv \int w(u, v; u', v') f(u') f(v')\, du\, du'\, dv'.$$

Therefore by (1.15)

$$C(f) = \int w(u, v; u', v') \left[f(u')f(v') - f(u)f(v)\right] du\, du'\, dv'. \tag{1.19}$$

Note that x is unchanged in $C(f)$.

For a monatomic gas, we write

$$\frac{w\, du'\, dv'}{|v - u|} = d\sigma \tag{1.20}$$

which is called the *differential collision cross section*. $d\sigma$ contains δ functions

$$\delta(u' + v' - u - v) \cdot \delta\left(\frac{|u'|^2 + |v'|^2 - |u|^2 - |v|^2}{2}\right)$$

expressing conservation of momentum and energy. Assume these have been removed. Then $d\sigma$ = *scattering cross section*. One usually writes this as $d\sigma = q(\omega, |u - v|)\, d\omega \quad (\omega \in S^2)$ so that

$$C(f) = \int_{\mathbb{R}^3} \int_{|\omega|=1} q(\omega, |u - v|)(f(u')f(v') - f(u)f(v))\, d\omega\, du. \tag{1.21}$$

Now we obtain the explicit form of u', v'. The conservation laws impose four constraints on the six variables u', v'. Thus there are two degrees of freedom. We write

$$\begin{aligned} u' &= u + a(u, v, \omega)\omega \\ v' &= v - a(u, v, \omega)\omega \end{aligned} \tag{1.22}$$

where a is a scalar function and $|\omega| = 1$. Then momentum is automatically conserved. Next we force energy conservation as in (1.14):

$$|u'|^2 + |v'|^2 = |u|^2 + a^2 + 2a\omega \cdot u + |v|^2 + a^2 - 2a\omega \cdot v = |u|^2 + |v|^2. \tag{1.23}$$

Therefore

$$a^2 = a(\omega \cdot v - \omega \cdot u)$$

and hence, as long as $a \neq 0$,

$$a(u, v, \omega) = \omega \cdot (v - u). \tag{1.24}$$

1.2.1. Special Cases of q.

(i) For the *hard sphere*:

$$q(\omega, |u - v|) = b_0 |u - v| |\cos\theta|$$

where b_0 is related to the size of the ball, $\omega \cdot (v - u) = |v - u| \cos\theta$ and $0 < \theta < \pi/2$.

(ii) For the *inverse power law*:

$$q(\omega, |v - u|) = |v - u|^{\gamma} |\cos\theta|^{-\gamma'} q_0(\theta),$$

where

$$s > 1, \ \gamma = 1 - \frac{4}{s}, \ \gamma' = 1 + \frac{2}{s}$$

$$q_0 \text{ is bounded, } q_0(\theta) \neq 0 \quad \text{near} \quad \theta = \pi/2.$$

1.3. The Hard Sphere Case

Now we consider in some detail the hard sphere case. Write

$$(1.25) \quad w(u, v; u', v') = \text{const.}\ \delta(u+v-u'-v') \cdot \delta\left((|u|^2 + |v|^2 - |u'|^2 - |v'|^2)/2\right)$$

where the constant is related to the size of the spheres. Thus $C(f)$ can be expressed via the formula

$$c \iint [f(v')f(u+v-v') - f(u)f(v)] \delta\left((|u|^2 + |v|^2 - |v'|^2 - |u+v-v'|^2)/2\right) du\, dv'.$$

Set $\lambda = v - v'$ so that

(1.26)

$$C(f) = c \iint [f(v-\lambda)f(u+\lambda) - f(u)f(v)] \delta\left((|u|^2 + |v|^2 - |v-\lambda|^2 - |u+\lambda|^2)/2\right) du\, d\lambda.$$

Write $d\lambda = \mu^2 d\mu\, d\omega = \mu^2 d\mu \sin\theta\, d\theta\, d\phi$ with the polar axis along $v - u$ so that $|\mu| \leq |v - u|$ and

$$\cos\theta = \frac{(v' - v) \cdot (v - u)}{|v' - v||v - u|}.$$

Denote by $\mathcal{A}$ the argument of the delta function above in (1.26). Then

$$(1.27) \quad \begin{aligned} 2\mathcal{A} &= |u|^2 + |v|^2 - |v - \lambda|^2 - |u + \lambda|^2 \\ &= |u|^2 + |v|^2 - (|v|^2 + |\lambda|^2 - 2v \cdot \lambda) - (|u|^2 + |\lambda|^2 + 2u \cdot \lambda) \\ &= -2|\lambda|^2 + 2(v - u) \cdot \lambda \\ &= 2\mu|v - u|\cos\theta - 2\mu^2. \end{aligned}$$

Now we use the identity from Lemma 1.3.1 below:

$$(1.28) \qquad \delta\left((x - a)(x - b)\right) = \frac{1}{|a - b|}[\delta(x - a) + \delta(x - b)] \quad (a \neq b).$$

Thus

$$
\begin{aligned}
\delta(\mathcal{A}) &= \delta\left((|u|^2 + |v|^2 - |v'|^2 - |u + v - v'|^2)/2\right) \\
&= \delta(\mu^2 - \mu|v - u|\cos\theta) \\
&= \delta((\mu)(\mu - |v - u|\cos\theta)) \\
&= \frac{1}{|u - v||\cos\theta|}(\delta(\mu) + \delta(\mu - |u - v|\cos\theta)).
\end{aligned}
\tag{1.29}
$$

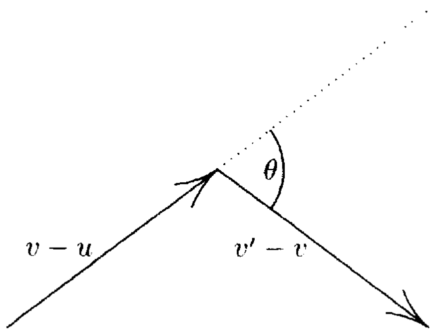

Fig. 1.2 *Geometry of Hard Sphere Collisions*

The $\delta(\mu)$-term gives 0 contribution because of the presence of the factor $\mu^2 d\mu$. The second term gives the contribution

$$
\frac{c}{|u - v||\cos\theta|} \cdot |u - v|^2 \cos^2\theta = |u - v||\cos\theta| \qquad (\text{as } \lambda = \mu\,\omega). \tag{1.30}
$$

Then in the integrand of (1.26)

$$
\begin{aligned}
&f(v - \lambda) = f(v - |v - u|\cos\theta\,\omega) = f(v - a\,\omega) = f(v'): \quad \text{similarly,} \\
&f(u + \lambda) = f(u + a\,\omega) \equiv f(u')
\end{aligned}
\tag{1.31}
$$

and therefore

$$
C(f) = c\int_{|\omega|=1}\int_{\mathbb{R}^3} |u - v||\cos\theta|(f(u')f(v') - f(u)f(v))\,d\omega\,du \tag{1.32}
$$

is the expression for $C(f)$ in the hard sphere case.

Now we establish the lemma used above.

LEMMA 1.3.1 *For $a \neq b$, $\phi \in \mathcal{D} \equiv C_0^\infty(\mathbb{R})$,*

$$
(\delta((x - a)(x - b)), \phi) = \frac{1}{|a - b|}(\phi(a) + \phi(b)). \tag{1.33}
$$

Proof. Let $\delta_\varepsilon \to \delta$ in $\mathcal{D}'$ with $\delta_\varepsilon \in L^1$. Then

$$I_\varepsilon \equiv (\delta_\varepsilon((x-a)(x-b)), \phi) = \int \delta_\varepsilon((x-a)(x-b))\phi(x)\,dx. \tag{1.34}$$

Partition this integral as

$$I_\varepsilon = I'_\varepsilon + I''_\varepsilon$$

where

$$I'_\varepsilon = \int_{-\infty}^{\frac{a+b}{2}} \delta_\varepsilon((x-a)(x-b))\phi(x)\,dx,$$

$$I''_\varepsilon = \int_{\frac{a+b}{2}}^{\infty} \delta_\varepsilon((x-a)(x-b))\phi(x)\,dx.$$

Without loss of generality we may assume that $a < b$. Set

$$y = x^2 - (a+b)x + ab = (x-a)(x-b) \quad \text{for} \quad x \le \frac{a+b}{2}.$$

Then y is monotone for such x and $y > 0$ if $x < a$; $y < 0$ if $a < x < \frac{a+b}{2}$, and $dy = [2x - (a+b)]\,dx$. Solving for x, we get

$$2x = a + b \pm \sqrt{(a+b)^2 - 4(ab-y)} = a + b \pm \sqrt{4y + (a-b)^2}. \tag{1.35}$$

Thus $dy = \pm\sqrt{4y + (a-b)^2}\,dx$ and hence

$$\begin{aligned} I'_\varepsilon &\equiv \int_{\infty}^{-\frac{(b-a)^2}{4}} \delta_\varepsilon(y) \frac{\phi\left(\frac{a+b-\sqrt{4y+(a-b)^2}}{2}\right) dy}{-\sqrt{4y+(a-b)^2}} \\ &\to \frac{\phi\left(\frac{a+b-|a-b|}{2}\right)}{|b-a|} = \frac{\phi(a)}{|b-a|}. \end{aligned}$$

We have taken the minus sign because y is decreasing on $x \le \frac{a+b}{2}$. Similarly,

$$\begin{aligned} I''_\varepsilon &= \int_{-\frac{(b-a)^2}{4}}^{\infty} \delta_\varepsilon(y) \frac{\phi\left(\frac{a+b+\sqrt{4y+(a-b)^2}}{2}\right) dy}{+\sqrt{4y+(a-b)^2}} \\ &\to \frac{\phi\left(\frac{a+b+|a-b|}{2}\right)}{|b-a|} = \frac{\phi(b)}{|b-a|}. \end{aligned}$$

1.4. Conservation Laws and the Entropy

Write

$$Q(f,f)(v) = \int_{\mathbb{R}^3}\int_{|\omega|=1} q(\omega, |u-v|)[f(u')f(v') - f(u)f(v)]\,du\,d\omega. \tag{1.36}$$

Define a related quadratic form as

(1.37)
$$Q^*(f,g)(v) = \frac{1}{2}\int_{\mathbb{R}^3}\int_{|\omega|=1} q[f(v')g(u')+f(u')g(v')-f(u)g(v)-f(v)g(u)]\,d\omega\,du.$$

Then Q^* is symmetric and $Q^*(f,f) = Q(f,f)$.

LEMMA 1.4.1 *For all smooth functions $f(v)$, $g(v)$, $\phi(v)$, small at infinity,*

$$\begin{aligned}
&\int Q^*(f,g)\phi(v)\,dv \\
&= \frac{1}{2}\iiint q[f(v')g(u') + f(u')g(v') - f(u)g(v) - f(v)g(u)]\phi(v)\,d\omega\,du\,dv \\
&= \frac{1}{2}\iiint q[f(v')g(u') + f(u')g(v') - f(u)g(v) - f(v)g(u)]\phi(u)\,d\omega\,du\,dv \\
&= \frac{1}{2}\iiint q[f(v')g(u') + f(u')g(v') - f(u)g(v) - f(v)g(u)](-\phi(v'))\,d\omega\,du\,dv \\
&= \frac{1}{2}\iiint q[f(v')g(u') + f(u')g(v') - f(u)g(v) - f(v)g(u)](-\phi(u'))\,d\omega\,du\,dv.
\end{aligned}$$

Proof. The first equation is the definition. Switch u, v: q is invariant since $q = q(\omega, |u-v|)$, and

(1.38)
$$\begin{aligned}
u' &= u + (\omega\cdot(v-u))\omega \to v + (\omega\cdot(u-v))\omega = v'; \\
v' &= v - (\omega\cdot(v-u))\omega \to u - (\omega\cdot(u-v))\omega = u'.
\end{aligned}$$

Therefore the second equation is true.

Now in the first equation change variables $(u,v)\mapsto(u',v')$:

$$\begin{aligned}
&\int Q^*(f,g)\phi(v)\,dv \\
&= \frac{1}{2}\iiint q(\omega,|v'-u'|)\phi(v(u',v'))|J|\Big[f(v')g(u') + f(u')g(v') \\
&\quad - f(u(u',v'))g(v(u',v')) - f(v(u',v'))g(u(u',v'))\Big]\,d\omega\,du'\,dv'.
\end{aligned}$$

We will show below that the absolute value of the Jacobian determinant $|J|$ is unity. Moreover, $|u-v|^2 = 2|u|^2 + 2|v|^2 - |u+v|^2 = |u'-v'|^2$ so q is invariant.

By definition, $v' = v - (\omega\cdot(v-u))\omega$; $u' = u + (\omega\cdot(v-u))\omega$. Therefore $v' - u' = v - u - 2(\omega\cdot(v-u))\omega$ and thus

(1.39)
$$\omega\cdot(v'-u') = \omega\cdot(v-u) - 2(\omega\cdot(v-u)) = -\omega\cdot(v-u).$$

We can invert these to get

(1.40)
$$\begin{aligned}
v &= v' + (\omega\cdot(v-u))\omega = v' - (\omega\cdot(v'-u'))\omega \equiv v(u',v'). \\
u &= u' - (\omega\cdot(v-u))\omega = u' + (\omega\cdot(v'-u'))\omega \equiv u(u',v').
\end{aligned}$$

Now rename (u', v') to (u, v) to get

$$\begin{aligned}
&\int Q^*(f,g)\phi(v)\,dv \\
&= \frac{1}{2}\iiint q(\omega, |v-u|)[f(v)g(u) + f(u)g(v) \\
&\qquad - f(u + \omega\cdot(v-u)\omega)g(v - \omega\cdot(v-u)\omega) \\
&\qquad - f(v - \omega\cdot(v-u)\omega)g(u + \omega\cdot(v-u)\omega)]\phi(v - (\omega\cdot(v-u))\omega)\,d\omega\,du\,dv \\
&= \frac{1}{2}\iiint q(\omega, |v-u|)[f(v)g(u) + f(u)g(v) \\
&\qquad - f(u')g(v') - f(v')g(u')]\phi(v')\,d\omega\,du\,dv
\end{aligned}$$

which is the third equation. For the last equation, switch u, v in the third equation; again $u' \to v', v' \to u'$ as above. This proves the Lemma.

Now take $g = f$ in Lemma 1.4.1 and add the four choices to get

$$\begin{aligned}
&\int Q(f,f)\phi(v)\,dv \\
&= \frac{1}{4}\iiint q(f(u')f(v') - f(u)f(v))[\phi(v) + \phi(u) - \phi(v') - \phi(u')]\,d\omega\,du\,dv.
\end{aligned}$$

Therefore

$$\int Q(f,f)\phi(v)\,dv = 0 \quad \text{if} \quad \phi(u') + \phi(v') = \phi(u) + \phi(v).$$

Such ϕ's are called **collisional (summational) invariants**. In particular, we can take

$$\phi(v) \equiv 1,\ \phi(v) = v_j \quad (j = 1, 2, 3);\ \phi(v) = |v|^2. \tag{1.41}$$

Corollary 1 : *$\int Q(f,f)\,dv = \int v_j Q(f,f)\,dv = \int |v|^2 Q(f,f)\,dv = 0$ for $j = 1, 2, 3$.*

Hence for a solution f to the (BE), suitably small at ∞, we have formally

$$\begin{aligned}
\iint f\,dv\,dx &= \text{const.} &&\text{(mass conservation)}\\
\iint v_j f\,dv\,dx &= \text{const.} &&\text{(momentum conservation)}\\
\iint |v|^2 f\,dv\,dx &= \text{const.} &&\text{(energy conservation).}
\end{aligned}$$

For this we simply choose ϕ as above and multiply the (BE) by $\phi(v)$: $\phi(v)f_t + \phi(v)v\cdot\nabla_x f = \phi(v)Q(f,f)$ and integrate.

Next, we assert the **Claim**: for f of restricted growth, $\int Q(f,f)\phi(v)\,dv = 0$ if and only if there exist $a \in \mathbb{R}$, $c < 0$ and $b \in \mathbb{R}^3$ such that $\phi(v) = a + b\cdot v + c|v|^2$. We will give the proof later. Hence, the only collisional invariants are as above. In [31] this result is called the "Boltzmann–Gronwall Theorem."

COROLLARY 2 : *(Entropy; the H-Theorem) Assume $q \geq 0$, $f > 0$. Then*

$$\frac{d}{dt}\iint f \ln \frac{1}{f}\,dv\,dx \geq 0. \tag{1.42}$$

Proof. Add the four choices in Lemma 1.4.1 and let $\phi = 1 + \ln f$:

$$\begin{aligned}
4\int Q(f,f)(1+\ln f)\,dv &= \iiint q[f(v')f(u') - f(u)f(v)]\cdot \\
&\quad [\ln f(v) + \ln f(u) - \ln f(v') - \ln f(u')]\,d\omega\,du\,dv \\
&= \iiint q[f(v')f(u') - f(u)f(v)]\cdot\{\ln f(u)f(v) - \ln f(u')f(v')\}\,d\omega\,du\,dv \\
&= \iiint q[f(v')f(u') - f(u)f(v)]\ln\frac{f(u)f(v)}{f(u')f(v')}\,d\omega\,du\,dv \\
&= \iiint qf(v')f(u')[1-\mu]\ln\mu\,d\omega\,du\,dv
\end{aligned}$$

where $\mu = \dfrac{f(u)f(v)}{f(u')f(v')}$. Now $(1-\mu)\ln\mu \leq 0$ for all $\mu > 0$. Thus

$$\int Q(f,f)\ln f\,dv \leq 0$$

for $f > 0$. Therefore

$$\begin{aligned}
\frac{d}{dt}\iint f\ln f\,dv\,dx &= \iint f_t(1+\ln f)\,dv\,dx \\
&= \iint(-v\cdot\nabla_x f + Q(f,f))(1+\ln f)\,dv\,dx \\
&= \iint Q(f,f)\ln f\,dv\,dx \leq 0
\end{aligned} \tag{1.43}$$

as desired.

The expression $-f\ln f$ is the *entropy density*.

1.5. Relevance of the Maxwellian

Now suppose that $\int Q(f,f)\cdot\ln f\,dv = 0$. Since the integrand is of one sign, we have

$$\ln f(u) + \ln f(v) = \ln f(u') + \ln f(v') \qquad a.e. \tag{1.44}$$

Thus if f is positive and continuous

$$\phi = \ln f \quad \text{satisfies} \quad \phi(u) + \phi(v) = \phi(u') + \phi(v'). \tag{1.45}$$

Hence

$$\phi(v) = a + b \cdot v + c|v|^2 \tag{1.46}$$

and thus f is a Gaussian: $f(v) = \exp(a + b \cdot v + c|v|^2) \quad (c < 0)$.

Here is a variational interpretation. Since $\iint f \ln f \, dv \, dx$ decreases in t, and

$$\iint f \, dv \, dx, \quad \iint vf \, dv \, dx, \quad \iint |v|^2 f \, dv \, dx$$

are constants, the system should tend to

$$\inf \iint f \ln f \, dv \, dx \tag{1.47}$$

subject to the 3 constraints. Use a, b, c as Lagrange multipliers and compute the Euler equation for the functional in (1.47). The result is

$$\frac{\partial}{\partial f}(f \ln f) - (a + b \cdot v + c|v|^2) = 0 \tag{1.48}$$

i.e.,

$$\ln f(v) = a' + b \cdot v + c|v|^2 \tag{1.49}$$

or

$$f(v) = \exp(a' + b \cdot v + c|v|^2) \quad (c < 0). \tag{1.50}$$

Again f is a Gaussian.

A direct lower bound may be achieved if the problem is restricted to a bounded domain in position space. Let $B \subseteq \mathbb{R}^3$ be a bounded set, say a cube, and consider periodic boundary conditions. Assume that

$$0 < f \in L^1(B \times \mathbb{R}^3), \quad |v|^2 f \in L^1(B \times \mathbb{R}^3).$$

Define

$$I_\lambda[f] = \iint_{B \times \mathbb{R}^3} f(\ln f + \lambda |v|^2) \, dv \, dx.$$

Let

$$M = \iint_{B \times \mathbb{R}^3} f \, dv \, dx,$$

$$\sigma(s) = \begin{cases} s \ln s & s > 0 \\ 0 & s = 0 \end{cases}.$$

Thus

$$I_\lambda[f] = \iint_{B\times\mathbb{R}^3} (\sigma(f) + \lambda|v|^2 f)\, dv\, dx.$$

Further, define

$$\mu_\lambda(v) = c_\lambda \exp(-\lambda|v|^2) \quad \text{where} \quad \iint_{B\times\mathbb{R}^3} \mu_\lambda\, dv\, dx = M.$$

It is easy to establish the following **Claim:**
For all $s > 0$ and all $s_0 > 0$, there exists a point ξ between them such that

$$\sigma(s) \geq s_0 \ln s_0 + \sigma'(s_0)\cdot(s - s_0) + \tfrac{1}{2}\frac{(s-s_0)^2}{s+s_0}.$$

Indeed by Taylor's Theorem we have

$$\begin{aligned}\sigma(s) &= \sigma(s_0) + \sigma'(s_0)\cdot(s-s_0) + \tfrac{1}{2}\sigma''(\xi)\cdot(s-s_0)^2\\ &= s_0\ln s_0 + (1+\ln s_0)\cdot(s-s_0) + \tfrac{1}{2}\frac{(s-s_0)^2}{\xi}.\end{aligned}$$

Now $\min\{s, s_0\} < \xi < \max\{s, s_0\}$ so clearly $\xi < s + s_0$. This proves the claim.

Moreover, we see that this inequality remains true for $s = 0,\ s_0 > 0$.

Applying the claim, we have

$$\begin{aligned}\sigma(f) &\geq \mu_\lambda \ln\mu_\lambda + (1+\ln\mu_\lambda)(f-\mu_\lambda) + \tfrac{1}{2}\frac{(f-\mu_\lambda)^2}{f+\mu_\lambda}\\ &= \mu_\lambda\ln\mu_\lambda + (1+\ln c_\lambda - \lambda|v|^2)(f-\mu_\lambda) + \tfrac{1}{2}\frac{(f-\mu_\lambda)^2}{f+\mu_\lambda}.\end{aligned}$$

Hence

$$\sigma(f) + \lambda|v|^2 f \geq \mu_\lambda\ln\mu_\lambda + \lambda|v|^2\mu_\lambda + (1+\ln c_\lambda)(f-\mu_\lambda) + \tfrac{1}{2}\frac{(f-\mu_\lambda)^2}{f+\mu_\lambda}.$$

Integrating over $B\times\mathbb{R}^3$ we get

$$I_\lambda[f] \geq I_\lambda[\mu_\lambda] + 0 + \tfrac{1}{2}\iint_{B\times\mathbb{R}^3}\frac{(f-\mu_\lambda)^2}{f+\mu_\lambda}\, dv\, dx.$$

We thus have the following result: Let

$$0 < f \in L^1(B\times\mathbb{R}^3), \quad |v|^2 f \in L^1(B\times\mathbb{R}^3).$$

Then the functional $I_\lambda[f]$ is minimized when f is the Gaussian μ_λ as above, normalized by the constant c_λ.

The above lower bound on $I_\lambda[f]$ leads to the following simple L^1 estimate on the difference $f - \mu_\lambda$:

$$\begin{aligned}\iint_{B\times\mathbb{R}^3} |f-\mu_\lambda|\,dv\,dx &= \iint_{B\times\mathbb{R}^3} |f+\mu_\lambda|^{\frac{1}{2}} \cdot \frac{|f-\mu_\lambda|}{|f+\mu_\lambda|^{\frac{1}{2}}}dv\,dx \\ &\le \left[\iint_{B\times\mathbb{R}^3} |f+\mu_\lambda|\,dv\,dx\right]^{\frac{1}{2}} \cdot \left[\iint_{B\times\mathbb{R}^3} \frac{(f-\mu_\lambda)^2}{|f+\mu_\lambda|}dv\,dx\right]^{\frac{1}{2}} \\ &\le 2M^{\frac{1}{2}} \cdot [I_\lambda[f] - I_\lambda[\mu_\lambda]]^{\frac{1}{2}} \\ &\le 2M^{\frac{1}{2}} \cdot [I_\lambda[f(0)] - I_\lambda[\mu_\lambda]]^{\frac{1}{2}}.\end{aligned}$$

Thus we have L^1-stability provided existence can be established in the appropriate sense. The above exposition of this well-known result is due to J. Schaeffer.

1.6. The Jacobian determinant

LEMMA 1.6.1 *The Jacobian matrix J satisfies*

$$\det J \equiv \left|\frac{\partial(u',v')}{\partial(u,v)}\right| = -1. \tag{1.51}$$

Proof. Write

$$u' = u + a\omega, \quad v' = v - a\omega \quad \text{where} \quad a = \omega\cdot(v-u). \tag{1.52}$$

Write J in 3×3 blocks:

$$J = \begin{bmatrix} \frac{\partial v'_i}{\partial v_j} & \frac{\partial u'_i}{\partial v_j} \\ \frac{\partial v'_i}{\partial u_j} & \frac{\partial u'_i}{\partial u_j} \end{bmatrix} = \begin{bmatrix} \delta_{ij} - \omega_i a_{v_j} & \omega_i a_{v_j} \\ -\omega_i a_{u_j} & \delta_{ij} + \omega_i a_{u_j} \end{bmatrix}. \tag{1.53}$$

Now add the fourth column to the first column, add the fifth column to the second column, and add the sixth column to the third column. Then

$$|J| = \det\begin{bmatrix} I & \omega_i a_{v_j} \\ I & \delta_{ij} + \omega_i a_{u_j} \end{bmatrix}. \tag{1.54}$$

Next subtract the first row from the fourth row, subtract the second row from the fifth row, and subtract the third row from the sixth row. There results

$$|J| = \det\begin{bmatrix} I & \omega_i a_{v_j} \\ 0 & \delta_{ij} + \omega_i b_j \end{bmatrix} \tag{1.55}$$

where $b_j = a_{u_j} - a_{v_j}$. Thus $|J| = \det(\delta_{ij} + \omega_i b_j)$ which is now a 3×3 matrix computation. We compute $b_j = a_{u_j} - a_{v_j} = -2\omega_j$ so that

$$\begin{aligned}|J| &= \det(\delta_{ij} - 2\omega_i\omega_j) \\ &= \begin{vmatrix} 1-2\omega_1^2 & -2\omega_1\omega_2 & -2\omega_1\omega_3 \\ -2\omega_2\omega_1 & 1-2\omega_2^2 & -2\omega_2\omega_3 \\ -2\omega_3\omega_1 & -2\omega_3\omega_2 & 1-2\omega_3^2 \end{vmatrix} \\ &= (1-2\omega_1^2)[(1-2\omega_2^2)(1-2\omega_3^2) - 4\omega_2^2\omega_3^2] \\ &\quad + 2\omega_1\omega_2[-2\omega_1\omega_2(1-2\omega_3^2) - 4\omega_1\omega_2\omega_3^2] \\ &\quad - 2\omega_1\omega_3[4\omega_1\omega_2^2\omega_3 + 2\omega_1\omega_3(1-2\omega_2^2)].\end{aligned} \tag{1.56}$$

After a simple calculation, we get $|J| = -1$.

1.7. The Structure of Collision Invariants

LEMMA 1.7.1 *Let $x \in \mathbb{R}^n$. Let f be continuous at one point x_0 and satisfy*

$$f(x) + f(y) = f(x+y) \quad \textit{for all} \quad x, y.$$

Then $f(x) = \xi \cdot x$ for some constant $\xi \in \mathbb{R}^n$.

Proof. We claim that f is homogeneous of degree one:

$$f(\alpha x) = \alpha f(x) \quad \text{for all} \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}. \tag{1.57}$$

Indeed, assume this temporarily and consider any orthonormal basis $\{e_k\}_{k=1}^n$ for $\mathbb{R}^n$. Let $x \in \mathbb{R}^n$. Then

$$\begin{aligned} f(x) &= f\left(\sum_{k=1}^n (x \cdot e_k) e_k\right) = \sum_{k=1}^n f\left((x \cdot e_k)e_k\right) \\ &= \sum_{k=1}^n (x \cdot e_k) f(e_k) = x \cdot \sum_{k=1}^n e_k f(e_k) \equiv x \cdot \xi \end{aligned} \tag{1.58}$$

as desired.
It remains to establish (1.57). f is everywhere continuous by

$$f(0) = 0, \quad f(x+h) - f(x) = f(h) = f(x_0 + h) - f(x_0).$$

Next, for $p \in \mathbb{N}, f(\sum_{i=1}^p x_i) = \sum_{i=1}^p f(x_i)$. We take $x_i = x$ for all i to get $f(px) = pf(x)$. Now we put $p \to q \in \mathbb{N}, x \to \frac{1}{q}x : f(x) = qf(\frac{x}{q})$, or

$$f\left(\frac{x}{q}\right) = \frac{1}{q} f(x)$$

and hence

$$f\left(\frac{p}{q}x\right) = pf\left(\frac{x}{q}\right) = \frac{p}{q} f(x)$$

i.e.,

$$f(\alpha x) = \alpha f(x) \quad \text{for all} \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{Q}.\ \alpha > 0 \tag{1.59}$$

and thus for all $x \in \mathbb{R}^n$, $\alpha \in \mathbb{R}$, $\alpha > 0$. By hypothesis, $f(0) = 0$ and $f(-x) = -f(x)$. Hence

$$f(\alpha x) = \alpha f(x) \quad \text{for all} \quad x \in \mathbb{R}^n,\ \alpha \in \mathbb{R}. \tag{1.60}$$

LEMMA 1.7.2 *Let g be continuous and satisfy*

$$g(u') + g(v') = g(u) + g(v) \quad \textit{for all} \quad u, v \in \mathbb{R}^3. \tag{1.61}$$

Then there exist constants a, $c \in \mathbb{R}$ and a constant vector $b \in \mathbb{R}^3$ such that

$$g(v) = a + b \cdot v + c|v|^2. \tag{1.62}$$

Proof. This is basically the proof from [9]. By hypothesis,

$$g(u) + g(v) = F(|u|^2 + |v|^2, u + v)$$

for some function F. Define

$$g_\pm(v) = g(v) \pm g(-v); \tag{1.63}$$
$$F_\pm(|u|^2 + |v|^2, u + v) = F(|u|^2 + |v|^2, u + v) \pm F(|u|^2 + |v|^2, -u - v).$$

From above, $g(-u) + g(-v) = F(|u|^2 + |v|^2, -u - v)$. Hence

$$g_\pm(u) + g_\pm(v) = F_\pm(|u|^2 + |v|^2, u + v). \tag{1.64}$$

Clearly we have $g_\pm(-v) = \pm g_\pm(v)$, etc.

Put $u = -v$ in (1.64):

$$\pm g_\pm(v) + g_\pm(v) = F_\pm(2|v|^2, 0). \tag{1.65}$$

Thus

$$2g_+(v) = F_+(2|v|^2, 0) \tag{1.66}$$

so $g_+(v)$ depends only on $|v|^2$. Write $g_+(v) = \zeta(|v|^2)$. From (1.64) then, F_+ depends only on $|u|^2 + |v|^2$ (see the observation below), and hence

$$\zeta(|v|^2) + \zeta(|u|^2) = F_+(|u|^2 + |v|^2). \tag{1.67}$$

Set $u = 0$ here to get $\zeta(|v|^2) + \zeta(0) = F_+(|v|^2)$. Hence

$$\zeta(|v|^2 + |u|^2) + \zeta(0) = F_+(|u|^2 + |v|^2) = \zeta(|v|^2) + \zeta(|u|^2).$$

Set $f(|v|^2) = \zeta(|v|^2) - \zeta(0)$. By Lemma 1.7.1, $f(|v|^2) = c|v|^2$ for some constant c. Thus

$$g_+(v) = \zeta(|v|^2) = f(|v|^2) + \zeta(0) = c|v|^2 + \zeta(0). \tag{1.68}$$

as desired.

For the function g_- we have

$$g_-(v) + g_-(u) = F_-(|u|^2 + |v|^2, u + v). \tag{1.69}$$

First we take u perpendicular to v; then $|u+v|^2 = |u|^2 + |v|^2$ so F_- depends on the second argument only. We can write

$$g_-(v) + g_-(u) = h(u+v). \tag{1.70}$$

Set $u = 0$:

$$g_-(v) + g_-(0) = h(v) \tag{1.71}$$

or

$$g_-(v) = h(v)$$

so that

$$g_-(v) + g_-(u) = h(u+v) = g_-(u+v).$$

Therefore we are done if u is perpendicular to v.

To avoid this extra hypothesis: let u, v be arbitrary, choose a vector ρ such that

$$\rho \cdot u = \rho \cdot v = 0, \quad |\rho|^2 = |u \cdot v| \qquad (\text{e.g.}, \quad \rho = \pm c\, u \times v) \tag{1.72}$$

By the above considerations,

$$\begin{aligned} g_-(v+\rho) &= g_-(v) + g_-(\rho) \\ g_-(u \mp \rho) &= g_-(u) \mp g_-(\rho). \end{aligned} \tag{1.73}$$

The sign in the second expression is chosen as follows: we take the minus sign if $u \cdot v > 0$, the plus sign if $u \cdot v < 0$.

Now

$$\begin{aligned} (v+\rho)\cdot(u \mp \rho) &= v \cdot u \mp v \cdot \rho + \rho \cdot u \mp |\rho|^2 \\ &= v \cdot u \mp |\rho|^2 = 0 \end{aligned} \tag{1.74}$$

by this choice of sign. Therefore

$$\begin{aligned} g_-(v+\rho) + g_-(u \mp \rho) &= g_-(v + u + \rho \mp \rho) \\ &= g_-(v+u) + g_-(\rho \mp \rho) \quad \begin{cases} \text{because trivially} \\ (v+u) \perp (\rho \mp \rho). \end{cases} \end{aligned} \tag{1.75}$$

By using (1.73) on the left-hand side, we get

$$g_-(v) + g_-(\rho) + g_-(u) \mp g_-(\rho) = g_-(v+u) + g_-(\rho \mp \rho). \tag{1.76}$$

If $u \cdot v > 0$, we take the minus sign: $g_-(v) + g_-(u) = g_-(u+v)$. Put $u = v$: $2g_-(v) = g_-(2v)$. Hence

$$2g_-(\rho) = g_-(2\rho). \tag{1.77}$$

Put this in (1.76) using the bottom sign:

$$g_-(v) + 2g_-(\rho) + g_-(u) = g_-(u+v) + g_-(2\rho). \tag{1.78}$$

Therefore

$$g_-(v) + g_-(u) = g_-(u+v) \tag{1.79}$$

so that

$$g_-(v) = b \cdot v \quad \text{for some constant vector} \quad b.$$

Above we used the following observation: in order to deduce (1.67) from (1.64), we show that no non-constant function of $u+v$ can be constructed from the arguments $|u|^2$ and $|v|^2$. For, suppose that

$$f(u+v) = g(|v|^2, |u|^2). \tag{1.80}$$

Put $u = 0$:

$$f(v) = g(|v|^2, 0) \equiv h(|v|^2) \tag{1.81}$$

hence

$$h(|u+v|^2) = h(|u|^2 + |v|^2 + 2u \cdot v). \tag{1.82}$$

Thus $f(u+v) = h(|u+v|^2)$ implies

$$h(|u|^2 + |v|^2 + 2u \cdot v) = g(|v|^2, |u|^2). \tag{1.83}$$

When $u \cdot v = 0$ and $|u|^2 = t|v|^2$ we have

$$\text{the left-hand side} = h((1+t)|v|^2);$$
$$\text{the right-hand side} = g(|v|^2, t|v|^2).$$

However, when $u = t^{1/2} v$,

$$\text{the left-hand side} = h(t|v|^2 + |v|^2 + 2t^{1/2}|v|^2) = h((t^{1/2}+1)^2|v|^2),$$
$$\text{the right-hand side} = g(|v|^2, t|v|^2).$$

So for these two different choices of arguments, the right-hand sides are equal, but the left-hand sides are not. Hence h must be constant.

The case in which the function g is merely measurable can be found in [10].

1.8. Relationship of the Boltzmann Equation to the Equations of Fluids

Suppose $f = f(t, x, v)$ is a solution to the (BE). Consider the following "translation table" in which we abbreviate $j_k = J_k(t, x) - v_k$:

Expression	*Name*	*Description*
$\int f\, dv$	$\rho(t, x)$	**density in physical space**
$\frac{\int vf\, dv}{\int f\, dv}$	$J(t, x)$	**mass velocity**
$\int vf\, dv$	$\rho J(t, x)$	**momentum density**
$\int j_i j_k f(t, x, v)\, dv$	$p_{ik}(t, x)$	**stress tensor**
$\frac{1}{2} \int v \cdot vf\, dv$		**energy density**
$\frac{1}{2\rho} \sum_i p_{ii}$	$e(t, x)$	**internal energy**
$-\frac{1}{2} \int j_k j \cdot j f\, dv$	$q_k(t, x)$	**heat flux vector**

Table 1.2.*Relationship to Fluids*

The **pressure** p corresponds to $\frac{1}{3} \sum_i p_{ii}$, so $p = \frac{2}{3}\rho e$. The equation $p = \frac{2}{3}\rho e$ is the **equation of state**. For a monatomic perfect gas, $e = e(T)$, $T =$ temperature. Thus by the above, $\frac{p}{\rho}$ is constant at constant temperatures. A **perfect gas** is given by $p = \rho RT$ $(R = \text{constant})$.

Now we use the Boltzmann conservation laws to get the fluid equations. The Boltzmann equation (BE) is

$$f_t + v \cdot \nabla_x f = Q(f, f). \tag{1.84}$$

Recall that $\int Q\, dv = \int v_j Q\, dv = \int |v|^2 Q\, dv = 0 \quad (j = 1, 2, 3)$.

Multiply the (BE) by 1 and integrate with respect to v:

$$\partial_t \int f\, dv + \nabla_x \cdot \int vf\, dv = 0, \tag{1.85}$$

or

$$\rho_t + \nabla_x \cdot (\rho J) = 0. \tag{1.86}$$

Multiply the (BE) by v_j $(j = 1, 2, 3)$ and integrate:

$$\partial_t \int v_j f\, dv + \nabla_x \cdot \int v_j v f\, dv = 0. \tag{1.87}$$

Now $\int v_j f\, dv = \rho J_j$ by definition and

$$\begin{aligned} \int v_j v_i f\, dv &= p_{ij} - J_i J_j \int f\, dv + J_j \int v_i f\, dv + J_i \int v_j f\, dv \\ &= p_{ij} - J_i J_j \rho + J_j \cdot \rho J_i + J_i \cdot \rho J_j \\ &= p_{ij} + \rho J_i J_j. \end{aligned} \tag{1.88}$$

Thus

$$\frac{\partial}{\partial t}(\rho J_j) + \sum_i \frac{\partial}{\partial x_i}(p_{ij} + \rho J_i J_j) = 0 \quad (j = 1, 2, 3). \tag{1.89}$$

Multiply the (BE) by $|v|^2$ and integrate:

$$\partial_t \int |v|^2 f\, dv + \nabla_x \cdot \int |v|^2 v f\, dv = 0. \tag{1.90}$$

Now

$$\begin{aligned} e(t,x) \equiv \frac{1}{2\rho}\sum_i p_{ii} &= \frac{1}{2\rho}\sum_i \int |J_i - v_i|^2 f\, dv \\ &= \frac{1}{2\rho}\int (|J|^2 + |v|^2 - 2v\cdot J) f\, dv \\ &= \frac{1}{2\rho}\left[|J|^2\rho + \int |v|^2 f\, dv - 2\rho J \cdot J\right] \\ &= \frac{1}{2\rho}\int |v|^2 f\, dv - \frac{1}{2}|J|^2. \end{aligned} \tag{1.91}$$

Therefore

$$\frac{1}{2}\int |v|^2 f\, dv = \rho e + \frac{1}{2}\rho |J|^2. \tag{1.92}$$

Hence

$$\partial_t \int |v|^2 f\, dv = \partial_t \left[|J|^2\rho + 2\rho e\right]. \tag{1.93}$$

Now to compute $\int |v|^2 v_i f\, dv$, write

$$\begin{aligned} -q_i &\equiv \frac{1}{2}\int (J_i - v_i)|J - v|^2 f\, dv \\ &= \frac{1}{2}\int (J_i - v_i)\left[|J|^2 + |v|^2 - 2\sum_j J_j v_j\right] f\, dv \\ &= \frac{1}{2}J_i\left[|J|^2\rho + (2\rho e + \rho|J|^2) - 2J\cdot\rho J\right] \\ &\qquad - \frac{1}{2}|J|^2 \int v_i f\, dv - \frac{1}{2}\int v_i |v|^2 f\, dv + \sum_j J_j \int v_i v_j f\, dv \\ &= J_i\rho e - \frac{1}{2}|J|^2\cdot\rho J_i - \frac{1}{2}\int v_i|v|^2 f\, dv + \sum_j J_j(p_{ij} + \rho J_i J_j) \\ &= J_i\rho e + \frac{1}{2}\rho|J|^2 J_i + \sum_j J_j p_{ij} - \frac{1}{2}\int v_i |v|^2 f\, dv. \end{aligned} \tag{1.94}$$

Thus

$$\partial_t\left[\rho\Big(e+\frac{1}{2}|J|^2\Big)\right]+\sum_i\frac{\partial}{\partial x_i}\left[\rho J_i\Big(e+\frac{1}{2}|J|^2\Big)+\sum_j J_j p_{ij}+q_i\right]=0. \tag{1.95}$$

The five equations (1.86), (1.89) and (1.95) are the five basic equations of continuum mechanics. These five equations have more than five unknowns. Indeed, ρ, e are two unknown scalars, while q, J are three vectors, which provide six unknowns. Finally, since p is a 3×3 symmetric matrix, we have an additional six unknowns, for a total of 14. However the known relation $\sum_i p_{ii}=2\rho e$ leaves us with 13 unknowns. The "magic" integer 13 will arise again in the near-equilibrium study of the Cauchy problem in Chapter 3. We quote from [31]: "The entire purpose of kinetic theory is to relate the 13 scalar fields ... to various circumstances of the kinetic gas." To make these consistent, we need to impose "constitutive equations" to relate p_{ij}, q_i to ρ, J_i, e.

1.8.1. Examples. As particular famous examples, we cite the following:

1. **The Euler Equations** (ideal fluids)

Take $p(t,x)$ to be a scalar-valued function, and

$$p_{ij}=p(t,x)\delta_{ij},\qquad q_i=0. \tag{1.96}$$

Then the classical Euler equations result.

2. **The Navier Stokes Equations** (viscous fluids)

Let $p(t,x)$ be as above and denote by μ, λ certain viscosity coefficients. One takes

$$\begin{aligned} p_{ij}&=p(t,x)\delta_{ij}-\mu\left(\frac{\partial J_i}{\partial x_j}+\frac{\partial J_j}{\partial x_i}\right)-\lambda\sum_k\frac{\partial J_k}{\partial x_k}\delta_{ij}\\ q_i&=-k\frac{\partial T}{\partial x_i}.\end{aligned} \tag{1.97}$$

Then the classical Navier Stokes equations result.

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Chapter 2

The Boltzmann Equation near the Vacuum

2.1. Invariance of $|x - tv|^2 + |x - tu|^2$

The goal in this chapter is to prove global existence to the Cauchy Problem for small (near vacuum) data in the hard sphere case. We follow [15] and [12]; see also [3]. For the density $f = f(t, x, v)$, $t \geq 0$; x, $v \in \mathbb{R}^3$, we write the Boltzmann equation (BE) as

$$f_t + v \cdot \nabla_x f = Q(f, f), \qquad f(0, x, v) = f_0(x, v) \tag{2.1}$$

where

$$\begin{aligned} Q(f, f) &= \sigma \int_{S_+^2} \int_{\mathbb{R}^3} \omega \cdot (v - u)[f(t, x, v')f(t, x, u') - f(t, x, u)f(t, x, v)]\, du\, d\omega \\ &= Q_g(f, f) - Q_\ell(f, f) \qquad = \text{“gain} - \text{loss”}. \end{aligned}$$

Here

$$\begin{aligned} S_+^2 &= \{\omega \in S^2 : \omega \cdot v \geq \omega \cdot u\} \\ \sigma &= \text{constant proportional to the area of the spheres} \\ \sigma^{-1} &= \text{a measure of the mean free path} \\ u' &= u + a\omega, \quad v' = v - a\omega, \quad a = \omega \cdot (v - u) \\ u' + v' &= u + v \qquad \text{(momentum conservation)} \\ |u'|^2 + |v'|^2 &= |u|^2 + |v|^2 \qquad \text{(energy conservation).} \end{aligned} \tag{2.2}$$

Write $Q_\ell(f, f) = fR(f)$ where

(2.3)

$$R(f)(t, x, v) = \sigma \int_{\mathbb{R}^3} \int_{S_+^2} \omega \cdot (v - u) f(t, x, u)\, du\, d\omega = \pi\sigma \int_{\mathbb{R}^3} |v - u| f(t, x, u)\, du.$$

Appropriate spaces for solution are as follows. Given $\beta > 0$, let

$$M = \Big\{ f \in C^0\left([0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3\right) : \quad \text{there exists } c > 0 \text{ such that}$$

$$|f(t,x,v)| \le ce^{-\beta(|x|^2+|v|^2)}\Big\}$$

with norm

$$\|f\| \equiv \sup_{t,x,v} e^{\beta(|x|^2+|v|^2)}|f(t,x,v)|.$$

We also define

$$X = \Big\{f : \quad f \text{ is measurable and there exists } c > 0 \text{ such that}$$

$$|f(t,x,v)| \le ce^{-\beta(|x|^2+|v|^2)} \quad \text{a.e.}\Big\}$$

with the same norm. We name the weight function:

$$\rho(x,v) = e^{\beta(|x|^2+|v|^2)}.$$

This stringent decay (which will be imposed on the initial data) greatly simplifies the argument but can be considerably weakened; see [15], [3].

We introduce the notation

$$f^\#(t,x,v) = f(t,x+vt,v).$$

Then the (BE) can be written as

$$\frac{d}{dt}f^\#(t,x,v) = Q^\#(f,f)(t,x,v). \tag{2.4}$$

Thus $Q^\#(f,f)(t,x,v) = Q_g^\#(f,f)(t,x,v) - Q_\ell^\#(f,f)(t,x,v)$ where

$$\begin{aligned}
Q_g^\#(f,f) &= \sigma\int_{S_+^2}\int_{\mathbb{R}^3} \omega\cdot(v-u)f(t,x+tv,v')f(t,x+tv,u')\,d\omega\,du\\
&= \sigma\int_{S_+^2}\int_{\mathbb{R}^3} \omega\cdot(v-u)f^\#(t,x+t(v-v'),v')f^\#(t,x+t(v-u'),u')\,d\omega\,du.\\
Q_\ell^\#(f,f) &= \pi\sigma f(t,x+tv,v)\int_{\mathbb{R}^3}|v-u|f(t,x+tv,u)\,du\\
&= \pi\sigma f^\#(t,x,v)\int_{\mathbb{R}^3}|v-u|f^\#(t,x+t(v-u),u)\,du.
\end{aligned}$$

It is the time integrated form of (2.4) to which we will find a continuous bounded nonnegative solution in this chapter.

Before presenting the details, we offer the following argument (due to Bardos, Degond and Golse [1]) which simply exposes the algebraic device upon which the proof rests. Given a *bounded* scattering kernel q, we introduce in the equation

$$f_t + v\cdot\nabla_x f = Q(f,f) \tag{2.5}$$

the change of variables $f = \mu F$, where $\mu = e^{-|x-tv|^2}$. Thus

$$\mu_t + v \cdot \nabla_x \mu = 0 \tag{2.6}$$

and

$$\mu_t F + \mu F_t + v \cdot (F\nabla_x \mu + \mu \nabla_x F) = Q(\mu F, \mu F) \tag{2.7}$$

or

$$\begin{aligned} F_t + v \cdot \nabla_x F &= \mu^{-1} Q(\mu F, \mu F) \\ &= \mu^{-1} \int\!\!\int q\left[\mu(v')F(v')\mu(u')F(u') - \mu(u)F(u)\mu(v)F(v)\right] d\omega\, du \end{aligned}$$

where as before we abbreviate $F(u) = F(t, x, u)$, etc. Now note that

$$\begin{aligned} \mu(u')\mu(v') &= \exp[-|x - tv'|^2 - |x - tu'|^2] \\ &= \exp[-|x|^2 - t^2|v'|^2 + 2tx \cdot v' - |x|^2 - t^2|u'|^2 + 2tx \cdot u'] \\ &= \exp[-|x - tv|^2 - |x - tu|^2] \\ &= \mu(u)\mu(v). \end{aligned} \tag{2.8}$$

Therefore

$$F_t + v \cdot \nabla_x F = \int\!\!\int q e^{-|x-tu|^2} (F(u')F(v') - F(u)F(v))\, d\omega\, du. \tag{2.9}$$

Since $0 \le q$ is bounded, we have for the right-side in (2.9) the upper bound

$$c(\sup |F(t)|)^2 \int e^{-|x-tu|^2} du \le \frac{c\|F(t)\|_\infty^2}{(1+t)^3} \tag{2.10}$$

(we are ignoring the possible singularity in the u-integral in (2.10) at $t = 0$). Under these conditions we have, with F_0 denoting data terms,

$$\|F(t)\|_\infty \le \|F_0(t)\|_\infty + c\int_0^t \frac{\|F(s)\|_\infty^2}{(1+s)^3}\, ds. \tag{2.11}$$

We set

$$\|F(t)\| = \sup_{0 \le s \le t} \|F(s)\|_\infty. \tag{2.12}$$

Then

$$\|F(t)\| \le c_0 + c_1 \|F(t)\|^2. \tag{2.13}$$

The inequality in (2.13) implies a bound on $\|F(t)\|$ provided that either c_0 or c_1 is sufficiently small. In order to see this, define

$$\Phi(F) = c_1 F^2 + c_0 - F. \tag{2.14}$$

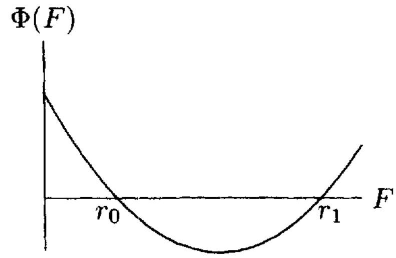

Fig. 2.1 *Small-data Convexity*

Notice that Φ is strictly convex. Assume that Φ has a negative minimum, as shown in Fig. 2.1. By (2.13), we will have either $0 < F \leq r_0$ or else $F \geq r_1$. By continuity, only the former case $0 < F \leq r_0$ will prevail if this inequality holds initially. In order to confirm this we compute directly

$$\Phi'(F) = 2c_1 F - 1 = 0 \quad \text{at} \quad F = \frac{1}{2c_1}, \tag{2.15}$$

so that

$$\Phi_{\min} = c_1 \cdot \frac{1}{4c_1^2} + c_0 - \frac{1}{2c_1} = -\frac{1}{4c_1} + c_0 < 0 \tag{2.16}$$

for c_0 (or c_1) small enough.

Turning to the proof of the major result, we begin with a calculus lemma.

LEMMA 2.1.1 *Let* $I = \int_0^\infty e^{-\beta|x+\tau(v-u)|^2}\, d\tau$. *Then*

$$I \leq \sqrt{\frac{\pi}{\beta}} \cdot \frac{1}{|v-u|}.$$

Proof.

$$\begin{aligned} I &= \int_0^\infty e^{-\beta|x+\tau(v-u)|^2}\, d\tau = \int_0^\infty e^{-\beta[|x|^2+\tau^2|v-u|^2+2\tau x\cdot(v-u)]}\, d\tau \\ &= e^{-\beta|x|^2} \int_0^\infty e^{-[\beta\tau^2|v-u|^2+2\tau\beta x\cdot(v-u)]}\, d\tau. \end{aligned}$$

With $s = |v-u|\tau$ we have

$$I = \frac{e^{-\beta|x|^2}}{|v-u|} \int_0^\infty e^{-\beta[s^2+2\frac{s}{|v-u|}x\cdot(v-u)]} ds.$$

Write $\eta = \frac{v-u}{|v-u|}$ so that $|\eta| = 1$. In the exponent here we have

$$\begin{aligned} s^2 + 2sx\cdot\eta &= s^2 + 2(x\cdot\eta)s + (x\cdot\eta)^2 - (x\cdot\eta)^2 \\ &= (s + (x\cdot\eta))^2 - (x\cdot\eta)^2. \end{aligned}$$

Thus

$$
\begin{aligned}
(2.17) \qquad I &= \frac{e^{-\beta|x|^2}}{|v-u|} \cdot e^{\beta(x\cdot\eta)^2} \int_0^\infty e^{-\beta(s+(x\cdot\eta))^2} ds \\
&\le \frac{1}{|v-u|} e^{-\beta|x\times\eta|^2} \int_{-\infty}^\infty e^{-\beta\xi^2} d\xi \le \sqrt{\frac{\pi}{\beta}} \cdot \frac{1}{|v-u|}
\end{aligned}
$$

as desired.

Next, we estimate the two parts of the collision operator, and show that the norm on M is "reproducing".

LEMMA 2.1.2 *For any $t \ge 0$ and $f\# \in M$ we have with a constant c independent of t*

$$
\begin{aligned}
\int_0^t |Q_g^\#(f,f)|\, d\tau &\le c\sigma\beta^{-2}\rho(x,v)^{-1}\|f\#\|^2 \\
\int_0^t |Q_\ell^\#(f,f)|\, d\tau &\le c\sigma\beta^{-2}\rho(x,v)^{-1}\|f\#\|^2.
\end{aligned}
$$

More generally, abbreviate $q = \omega\cdot(v-u)$ and, for $f_1^\#$, $f_2^\# \in M$, $t \ge 0$, define $\tilde{Q}_g^\#(f_1,f_2)$ and $\tilde{Q}_\ell^\#(f_1,f_2)$ as functions of (t,x,v) by

$$
\begin{aligned}
\tilde{Q}_g^\#(f_1,f_2) &= \sigma \int_{S_+^2}\int_{\mathbb{R}^3} q f_1^\#(t,x+t(v-v'),v') f_2^\#(t,x+t(v-u'),u')\, d\omega\, du \\
\tilde{Q}_\ell^\#(f_1,f_2) &= \pi\sigma f_1^\#(t,x,v) \int_{\mathbb{R}^3} |v-u| f_2^\#(t,x+t(v-u),u)\, du.
\end{aligned}
$$

Then

$$
\begin{aligned}
\int_0^t |\tilde{Q}_g^\#(f_1,f_2)|\, d\tau &\le c\sigma\beta^{-2}\rho(x,v)^{-1}\|f_1^\#\|\|f_2^\#\| \\
\int_0^t |\tilde{Q}_\ell^\#(f_1,f_2)|\, d\tau &\le c\sigma\beta^{-2}\rho(x,v)^{-1}\|f_1^\#\|\|f_2^\#\|.
\end{aligned}
$$

Proof. By definition

$$
\begin{aligned}
\left|Q_\ell^\#(f,f)(t,x,v)\right| &= \left|\pi\sigma f\#(t,x,v) \int_{\mathbb{R}^3} |v-u| f\#(t,x+t(v-u),u)\, du\right| \\
&\le \pi\sigma\rho(x,v)^{-1}\|f\#\|^2 \int_{\mathbb{R}^3} |v-u| e^{-\beta(|x+t(v-u)|^2+|u|^2)}\, du.
\end{aligned}
$$

Thus

$$
\begin{aligned}
&\int_0^t |Q_\ell^\#(f,f)|\, d\tau \\
&\le \pi\sigma\rho(x,v)^{-1}\|f\#\|^2 \int_{\mathbb{R}^3} |v-u| e^{-\beta|u|^2} \int_0^t e^{-\beta|x+\tau(v-u)|^2}\, d\tau\, du \\
&\le \pi\sigma\rho(x,v)^{-1}\|f\#\|^2 \int_{\mathbb{R}^3} |v-u| e^{-\beta|u|^2} \int_0^\infty e^{-\beta|x+\tau(v-u)|^2}\, d\tau\, du.
\end{aligned}
$$

The τ–integral we recognize as I, and thus from Lemma 2.1.1

$$\int_0^t |Q_\ell^\#(f,f)|\,d\tau \le \pi^{3/2}\sigma\rho(x,v)^{-1}\beta^{-1/2}\|f\#\|^2 \int_{\mathbb{R}^3} e^{-\beta|u|^2}\,du$$
$$= \pi^3\sigma\rho(x,v)^{-1}\beta^{-2}\|f\#\|^2$$

which is the desired estimate for the loss term. For the gain term, again with $q = \omega\cdot(v-u)$, we have

$$\begin{aligned}
&\int_0^t |Q_g^\#(f,f)|\,d\tau \\
&\le \sigma\int_0^t \left|\int_{S_+^2}\int_{\mathbb{R}^3} qf\#(\tau,x+\tau(v-v'),v')f\#(\tau,x+\tau(v-u'),u')\,d\omega\,du\right|\,d\tau \\
&\le \sigma\int_0^t\int_{S_+^2}\int_{\mathbb{R}^3} |q|\|f\#\|^2\exp\left[-\beta(|x+\tau(v-v')|^2+|v'|^2)\right]\cdot \\
&\qquad \exp\left[-\beta(|x+\tau(v-u')|^2+|u'|^2)\right]\,d\omega\,du\,d\tau \qquad (2.18)\\
&= \sigma\|f\#\|^2\int_{S_+^2}\int_{\mathbb{R}^3} |\omega\cdot(v-u)|\exp\left[-\beta(|v'|^2+|u'|^2)\right]\cdot \\
&\qquad \int_0^t \exp\left[-\beta|x+\tau(v-v')|^2\right]\exp\left[-\beta|x+\tau(v-u')|^2\right]\,d\tau\,d\omega\,du.
\end{aligned}$$

Denote by $\mathcal{P}$ the argument of the exponential function in the last line of (2.18). Then using the previous observation (essentially (2.8)) and the conservation laws in (2.2), we get

$$\begin{aligned}
-\frac{\mathcal{P}}{\beta} &\equiv |x+\tau(v-v')|^2+|x+\tau(v-u')|^2 \\
&= |(x+\tau v)-\tau v'|^2+|(x+\tau v)-\tau u'|^2 \qquad (2.19)\\
&= 2|x+\tau v|^2+\tau^2(|v'|^2+|u'|^2)-2\tau(x+\tau v)\cdot(v'+u') \\
&= 2|x+\tau v|^2+\tau^2(|v|^2+|u|^2)-2\tau(x+\tau v)\cdot(v+u) \\
&= |(x+\tau v)-\tau v|^2+|(x+\tau v)-\tau u|^2 \\
&= |x|^2+|x+\tau(v-u)|^2.
\end{aligned}$$

Now, using Lemma 2.1.1 and the conservation of energy, we find

$$\int_0^t |Q_g^\#(f,f)|\,d\tau \le \pi\sigma\|f\#\|^2\rho(x,v)^{-1}\int_{\mathbb{R}^3} |v-u|\cdot I\cdot\exp[-\beta|u|^2]\,du$$
$$\le \pi^3\beta^{-2}\sigma\rho(x,v)^{-1}\|f\#\|^2$$

which is the desired estimate for the gain term. The second part of this lemma is established in exactly the same manner.

Write $f(0,x,v)=f_0(x,v)$. Returning to (2.4), we integrate in time to get

$$f\#(t,x,v) = f_0(x,v) + \int_0^t Q\#(f,f)\,d\tau. \qquad (2.20)$$

Define the operator $\mathcal{F}$ on M by

$$\mathcal{F}f\# = f_0(x,v) + \int_0^t Q\#(f,f)\,d\tau$$

and let

$$M_R = \{f \in M : \|f\| \leq R\}.$$

LEMMA 2.1.3 *There exists a constant R_0 such that if $\|f_0\|$ and $\sigma\beta^{-2}R_0$ are sufficiently small, then the equation (2.20) has a unique solution $f\# \in M_{R_0}$. Moreover, under the same restrictions on f_0 and R_0, this equation is uniquely solvable in X as well.*

Proof. The estimates of Lemma 2.1.2 show that if e.g., $\|f_0\| \leq R/2$ and $f\# \in M_R$, then

$$\begin{aligned}|\mathcal{F}f\#| &\leq \rho(x,v)^{-1}\|f_0\| + 2\cdot\pi^3\beta^{-2}\sigma\rho(x,v)^{-1}\|f\#\|^2\\ &\leq \rho(x,v)^{-1}\left[\frac{R}{2} + 2\cdot\pi^3\beta^{-2}\sigma R^2\right].\end{aligned}$$

Thus $\mathcal{F}$ maps M_R into itself for R sufficiently small. Explicitly, we require that $2\cdot\pi^3\beta^{-2}\sigma R \leq 1/2$. Similarly, we show that $\mathcal{F}$ is a contraction on M_R for suitably small R. Since elements of M_R are continuous, the continuity of $\mathcal{F}f\#$ is evident.

2.2. Sequences of Approximate Solutions

It remains to show that the solution just obtained in the first part of Lemma 2.1.3 remains nonnegative. For this purpose we use the iteration of [13] and [12] as follows. Let $T > 0$ be arbitrary and let M_T denote the restriction of elements $f \in M$ to $[0,T] \times \mathbb{R}^3 \times \mathbb{R}^3$. Suppose that there exist $u_0^\#$, $\ell_0^\# \in M$ such that $\ell_0(t,x,v) \leq u_0(t,x,v)$ for all $0 \leq t < T$, $(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3$.

Define two sequences $\{\ell_k\}, \{u_k\}$ by

$$\begin{aligned}&\frac{d}{dt}\ell_{k+1}^\# + \ell_{k+1}^\# R\#(u_k) = Q_g^\#(\ell_k,\ell_k), \quad \ell_{k+1}(0) = f_0\\ &\frac{d}{dt}u_{k+1}^\# + u_{k+1}^\# R\#(\ell_k) = Q_g^\#(u_k,u_k), \; u_{k+1}(0) = f_0.\end{aligned} \tag{2.21}$$

Because we have assumed that $u_0^\# \in M$, the estimates of Lemma 2.1.2 allow us to conclude that

$$R\#(u_0),\, Q_g^\#(u_0,u_0) \in L^1((0,T), C^0(\mathbb{R}^3 \times \mathbb{R}^3)). \tag{2.22}$$

Clearly there exists a solution when $k = 0$. These are linear ordinary differential equations; thus if ℓ_{k-1}, u_{k-1} exist on $(0,T)$ then so do ℓ_k, u_k.

LEMMA 2.2.1 *Let $0 \leq f_0 \in M$. Assume the* **beginning condition** *(BC)*

$$0 \leq \ell_0(t) \leq \ell_1(t) \leq u_1(t) \leq u_0(t), \qquad 0 \leq t < T. \tag{2.23}$$

Then the system (2.21) *has a unique solution*

$$\ell_k^{\#},\ u_k^{\#} \in M_T \tag{2.24}$$

for all $k \geq 1$ *with the property*

$$\ell_{k-1}(t) \leq \ell_k(t) \leq u_k(t) \leq u_{k-1}(t), \qquad 0 \leq t < T. \tag{2.25}$$

Temporarily we assume that the (BC) and the result of the lemma hold with some $u_0^{\#} \in M_T$. Then there exist functions ℓ, u with $\ell_k \nearrow \ell$, $u_k \searrow u$, and $\ell(t) \leq u(t)$ for all t. Now integrate over $[0,t]$ the ordinary differential equations (2.21) at step k; let $k \to \infty$ and apply the dominated convergence theorem to get

$$\begin{aligned} \ell^{\#}(t) + \int_0^t \ell^{\#} R^{\#}(u)(\tau)\, d\tau &= f_0 + \int_0^t Q_g^{\#}(\ell,\ell)(\tau)\, d\tau \\ u^{\#}(t) + \int_0^t u^{\#} R^{\#}(\ell)(\tau)\, d\tau &= f_0 + \int_0^t Q_g^{\#}(u,u)(\tau)\, d\tau. \end{aligned} \tag{2.26}$$

This is the *separated Boltzmann system*. If we can show that $u = \ell$, then $f \equiv u = \ell$ will be a nonnegative "mild" solution of the Boltzmann equation (2.1).

Proof of Lemma 2.2.1. In order to see the monotonicity, we solve explicitly to get

$$\ell_k^{\#}(t) = f_0 e^{-\int_0^t R^{\#}(u_{k-1})ds} + \int_0^t e^{-\int_\tau^t R^{\#}(u_{k-1})ds} Q_g^{\#}(\ell_{k-1},\ell_{k-1})\, d\tau. \tag{2.27}$$

Thus

$$\ell_{k+1}^{\#}(t) = f_0 e^{-\int_0^t R^{\#}(u_k)ds} + \int_0^t e^{-\int_\tau^t R^{\#}(u_k)ds} Q_g^{\#}(\ell_k,\ell_k)\, d\tau. \tag{2.28}$$

Assume that for some $k \geq 1$

$$\ell_{k-1}(t) \leq \ell_k(t) \leq u_k(t) \leq u_{k-1}(t) \tag{2.29}$$

and subtract (2.27) from (2.28):

$$\begin{aligned} \ell_{k+1}^{\#}(t) - \ell_k^{\#}(t) &= f_0 \left[e^{-\int_0^t R^{\#}(u_k)ds} - e^{-\int_0^t R^{\#}(u_{k-1})ds} \right] \\ &+ \int_0^t \left[e^{-\int_\tau^t R^{\#}(u_k)ds} - e^{-\int_\tau^t R^{\#}(u_{k-1})ds} \right] Q_g^{\#}(\ell_k,\ell_k)\, d\tau \\ &+ \int_0^t e^{-\int_\tau^t R^{\#}(u_{k-1})ds} \left[Q_g^{\#}(\ell_k,\ell_k) - Q_g^{\#}(\ell_{k-1},\ell_{k-1}) \right] d\tau. \end{aligned} \tag{2.30}$$

The hard sphere kernel is nonnegative on the set of integration, and from definition (2.3), $R(u) \le R(v)$ if $u \le v$ a.e. So the first two terms are nonnegative. By the induction assumption, the last term is too, since Q_g is monotone. Hence

$$\ell_k^{\#}(t) \le \ell_{k+1}^{\#}(t) \tag{2.31}$$

and a similar argument applies to the $\{u_k^{\#}(t)\}$. We see that each member of $\{\ell_k^{\#}\}$, $\{u_k^{\#}\}$ is nonnegative and belongs to M_T by using the estimates of Lemma 2.1.2. This proves the Lemma.

In order to simplify the (BC), we take $\ell_0 = 0$ and any $0 \le u_0^{\#} \in M_T$. We claim that

$$0 = \ell_0(t) \le \ell_1(t) \le u_1(t). \tag{2.32}$$

Indeed, by the differential equations,

$$\begin{aligned} &\frac{d}{dt}\ell_1^{\#} + \ell_1^{\#} R\#(u_0) = Q_g^{\#}(\ell_0, \ell_0) \\ &\frac{d}{dt}u_1^{\#} + u_1^{\#} R\#(\ell_0) = Q_g^{\#}(u_0, u_0). \end{aligned} \tag{2.33}$$

Now

$$\ell_0 = 0 \text{ implies } R\#(\ell_0) = 0,\ Q_g^{\#}(\ell_0, \ell_0) = 0. \tag{2.34}$$

Therefore

$$\begin{aligned} u_1^{\#} &= f_0 + \int_0^t Q_g^{\#}(u_0, u_0)\, ds, \\ 0 \le \ell_1^{\#} &= f_0 e^{-\int_0^t R^{\#}(u_0)ds} \le f_0 \le u_1^{\#}. \end{aligned} \tag{2.35}$$

Hence the (BC) reduces to

$$u_1(t) \le u_0(t). \tag{2.36}$$

2.3. Satisfaction of the Beginning Condition

Write

$$\psi(v) = \sup_x e^{\beta|x|^2} f_0(x, v). \tag{2.37}$$

Since $0 \le f_0 \in M$,

$$\psi \le c_0 \exp(-\beta|v|^2). \tag{2.38}$$

LEMMA 2.3.1 *If*

$$\sigma\ \beta^{-2}R \quad \textit{and} \quad \|f_0\|$$

are sufficiently small, then the (BC) holds, and the separated Boltzmann system has a global solution (ℓ, u) *with* $(\ell\#, u\#) \in M_R$.

Proof. Since $\ell_0 = 0$,

$$u_1^\#(t) = f_0 + \int_0^t Q_g^\#(u_0, u_0)\, d\tau, \tag{2.39}$$

i.e., with $q = \omega \cdot (v - u)$,

(2.40)
$$u_1(t, x+tv, v) = f_0(x, v) + \sigma \int_0^t \int_{\mathbb{R}^3} \int_{S_+^2} q\, u_0(\tau, x+\tau v, v') u_0(\tau, x+\tau v, u')\, d\omega\, du\, d\tau.$$

We will look for u_0 in the form

$$u_0(t, x, v) = \tilde{v}(x - tv, v). \tag{2.41}$$

Thus the (BC) holds if

$$\begin{aligned} \text{(2.42)} \quad \sigma \int_0^t \int_{\mathbb{R}^3} \int_{S_+^2} & q\tilde{v}(x + \tau(v - v'), v')\tilde{v}(x + \tau(v - u'), u')\, d\omega\, du\, d\tau \\ & \leq \tilde{v}(x, v) - f_0(x, v). \end{aligned}$$

Set

$$u_0(t, x, v) = \tilde{v}(x - tv, v) = e^{-\beta|x - tv|^2} w(v) \tag{2.43}$$

i.e.,

$$\tilde{v}(x, v) = e^{-\beta|x|^2} w(v). \tag{2.44}$$

Then

$$\begin{aligned} \text{(2.45)} \quad & \tilde{v}(x + \tau(v - v'), v')\tilde{v}(x + \tau(v - u'), u') \\ & = w(v')w(u') \exp\left[-\beta|x + \tau(v - v')|^2 - \beta|x + \tau(v - u')|^2\right] \\ & = w(v')w(u') \exp\left[-\beta(|x|^2 + |x + \tau(v - u)|^2\right]. \end{aligned}$$

We have used (2.19) in the last step. Now we multiply (2.42) by $e^{\beta|x|^2}$. Thus the (BC) holds if

(2.46)
$$\psi(v) + \sigma \int_0^t \int_{\mathbb{R}^3} \int_{S_+^2} \omega \cdot (v - u) e^{-\beta|x + \tau(v-u)|^2} w(v')w(u')\, du\, d\omega\, d\tau \leq w(v).$$

We want a nonnegative solution w of (2.46). Using Lemma 2.1.1. we see that a sufficient condition for (2.46) to hold is

$$\psi(v) + \sigma\sqrt{\frac{\pi}{\beta}} \int_{S_+^2} \int_{\mathbb{R}^3} w(v')w(u')\, d\omega\, du = w(v). \tag{2.47}$$

To prove the existence of a solution $w \geq 0$ to (2.47) consider the space

$$G = \left\{ g \in C^0(\mathbb{R}^3) : \quad \text{there exists } c > 0 \text{ with } \quad |g(v)| \leq c \exp(-\beta |v|^2) \right\}$$

with norm

$$\|g\|_G = \sup_v \exp(\beta |v|^2) |g(v)|.$$

Consider the operator $\mathcal{T}$ defined on G by

$$\mathcal{T}(w)(v) = \psi(v) + \epsilon \int_{\mathbb{R}^3} \int_{S^2_+} w(v') w(u') \, du \, d\omega \tag{2.48}$$

where

$$\epsilon = \sigma \sqrt{\frac{\pi}{\beta}}. \tag{2.49}$$

We claim that $\mathcal{T}$ maps a sufficiently small ball in G into itself. Indeed, let $w \geq 0$. Then since $0 \leq f_0 \in M$, $\mathcal{T}(w)(v) \geq 0$ and

$$\begin{aligned}
|\mathcal{T}(w)(v)| &\leq c_0 \exp(-\beta |v|^2) + \epsilon \int_{\mathbb{R}^3} \int_{S^2_+} \|w\|_G^2 \exp\left[-\beta(|v'|^2 + |u'|^2)\right] d\omega \, du \\
&\leq \exp(-\beta |v|^2) \left(c_0 + 2\pi\epsilon \|w\|_G^2 \int_{\mathbb{R}^3} \exp(-\beta |u|^2) \, du \right) \\
&= \exp(-\beta |v|^2) \left(c_0 + 2\sigma\pi^3 \beta^{-2} \|w\|_G^2 \right).
\end{aligned}$$

Thus for such w

$$\|\mathcal{T} w\|_G \leq c_0 + 2\sigma\pi^3\beta^{-2} \|w\|_G^2.$$

Similarly, for $0 \leq w_1, w_2 \in G$.

$$\|\mathcal{T}(w_1) - \mathcal{T}(w_2)\|_G \leq 2\sigma\pi^3\beta^{-2} (\|w_1\|_G + \|w_2\|_G) \|w_1 - w_2\|_G. \tag{2.50}$$

Thus $\mathcal{T}$ maps nonnegative functions in the ball of radius R_0 in G into itself, and is a contraction there, if $\|f_0\|$ and $\sigma\beta^{-2}R_0$ are sufficiently small.

Since $\mathcal{T}$ is a contraction there we may write $w = \lim_{n\to\infty} \mathcal{T}^n(w)$. We may take $\mathcal{T}^0(w) = f_0$. Since $f_0 \geq 0$ by hypothesis, the solution to (2.47) is nonnegative. We have thus found the "starting" function u_0.

2.4. Proof that $u = \ell$

It remains to show that $u = \ell$. Take R from Lemma 2.3.1.

LEMMA 2.4.1 *When $\|f_0\|$ and $\sigma\beta^{-2}R$ are sufficiently small, $u = \ell$ where u, ℓ are the solutions of the separated Boltzmann system* (2.26).

Proof. By definition,

$$\ell\#(t) + \int_0^t \ell\# R\#(u)(\tau) \, d\tau = f_0 + \int_0^t Q_g^\#(\ell, \ell)(\tau) \, d\tau$$

$$u\#(t) + \int_0^t u\# R\#(\ell)(\tau) \, d\tau = f_0 + \int_0^t Q_g^\#(u, u)(\tau) \, d\tau.$$

Subtracting these equations, we have

$$(u\#-\ell\#)(t) = \int_0^t [Q_g^\#(u, u-\ell)+Q_g^\#(u-\ell, \ell)+\ell\# R\#(u-\ell)-(u\#-\ell\#)R\#(\ell)]\, d\tau.$$

Now we simply take norms in M, using the estimates from the second part of Lemma 2.1.2:

$$\|u\# - \ell\#\| \leq c\sigma\beta^{-2}\Big[\|u\#\|\|u\# - \ell\#\| + \|\ell\#\|\|u\# - \ell\#\|\Big].$$

Now $u\#$, $\ell\#$ both lie in M_R, so each of the factors $\|u\#\|$, $\|\ell\#\|$ is bounded by cR. The conclusion now follows when the product $\sigma\beta^{-2}R$ is sufficiently small.

As in Lemma 2.4.1, we can show that $u = \ell \in X$ under the same restrictions on σ, β and $\|f_0\|$. Thus the nonnegative solution just obtained must coincide with the unique solution $f \in X$ obtained from the last sentence of Lemma 2.1.3. Since $f_0 \in M$ by hypothesis, and since $M \subseteq X$, our solutions must be identical, and the solution f from Lemma 2.4.1 must remain nonnegative. Summarizing, we have

THEOREM 2.4.1 *Consider the integrated form of the Boltzmann equation (2.20). There exists a constants* c_0, R_0 *such that if* $\|f_0\| \leq c_0 R_0$ *and* $\sigma\beta^{-2}R_0$ *are sufficiently small, then equation (2.20) has a unique nonnegative solution* $f\# \in M_{R_0}$.

2.5. Remarks and Related Questions

Polewczak shows in [15] that similar results can be obtained for more general scattering kernels, and for a wider class of initial data (which demands less decay at infinity). See also [3] in this regard. Also in [15] classical solutions are obtained. These are smooth in x provided the data is sufficiently regular.

An open problem is to obtain such a global existence result for the *relativistic* Boltzmann equation. This equation is described in some detail in Chapter 3. Conservation of momentum remains the same

$$u' + v' = u + v$$

but conservation of energy requires

$$\sqrt{1+|u'|^2} + \sqrt{1+|v'|^2} = \sqrt{1+|u|^2} + \sqrt{1+|v|^2}$$

in contrast to the classical case

$$|u'|^2 + |v'|^2 = |u|^2 + |v|^2.$$

The appearance of square roots here causes the algebraic device (used e.g., in (2.8), (2.19)) to fail.

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Chapter 3

THE BOLTZMANN EQUATION NEAR THE EQUILIBRIUM

3.1. The Perturbation from Equilibrium

We begin by writing the Boltzmann equation (BE) as

$$F_t + v \cdot \nabla_x F = Q(F, F) \tag{3.1}$$

Let μ be a normalized Maxwellian

$$\mu = e^{-|v|^2}, \tag{3.2}$$

and set

$$F = \mu + \mu^{\frac{1}{2}} f. \tag{3.3}$$

The goal is to show that initial data which is a small perturbation of μ (and which vanishes at spatial infinity) launches a global solution. One studies the linearized equation for f first and hopes that it is dissipative (this would be analogous to the entropy inequality). There is indeed dissipation, but not on the entire space: there is a five-dimensional space (the null space of the operator L below) which may prohibit decay in time. For this reason, an exceptionally careful analysis of the linearized problem is necessary. Now we compute the equation for f:

$$\begin{aligned} \mu^{\frac{1}{2}} f_t + \mu^{\frac{1}{2}} v \cdot \nabla_x f &= Q^*\left(\mu + \mu^{\frac{1}{2}} f, \mu + \mu^{\frac{1}{2}} f\right) \\ &= 2Q^*\left(\mu, \mu^{\frac{1}{2}} f\right) + Q^*\left(\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} f\right) \end{aligned} \tag{3.4}$$

because

$$Q^*(\mu, \mu) = 0. \tag{3.5}$$

(Recall that Q^* is defined in Chapter 1). Now

$$\begin{aligned} 2Q^*(\mu, \mu^{\frac{1}{2}} f) &= \iint q[\mu(v')\mu^{\frac{1}{2}}(u')f(u') + \mu(u')\mu^{\frac{1}{2}}(v')f(v') \\ &\qquad - \mu(v)\mu^{\frac{1}{2}}(u)f(u) - \mu(u)\mu^{\frac{1}{2}}(v)f(v)]\, d\omega\, du \\ &= \iint q\mu^{\frac{1}{2}}(v)\mu^{\frac{1}{2}}(u)[\mu^{\frac{1}{2}}(v')f(u') + \mu^{\frac{1}{2}}(u')f(v') \\ &\qquad - \mu^{\frac{1}{2}}(v)f(u) - \mu^{\frac{1}{2}}(u)f(v)]\, d\omega\, du. \end{aligned}$$

Thus the equation for the perturbation f is

$$f_t + v \cdot \nabla_x f + L[f] = \Gamma(f, f) \tag{3.6}$$

where

$$\begin{aligned} \Gamma(f,f) &= \mu^{-\frac{1}{2}}(v) Q(\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} f) \\ L[f] &= -2\mu^{-\frac{1}{2}}(v) Q^*(\mu, \mu^{\frac{1}{2}} f), \quad \text{or} \\ L[f] &= \nu(v) f - Kf, \\ K &= K_2 - K_1 \end{aligned} \tag{3.7}$$

with

$$\begin{aligned} \nu(v) &= \iint q(\omega, |v-u|)\mu(u)\, du\, d\omega \equiv \textbf{the collision frequency}, \\ K_1 f &= \mu^{\frac{1}{2}}(v) \iint q(\omega, |v-u|) \mu^{\frac{1}{2}}(u) f(u)\, du\, d\omega, \\ K_2 f &= \iint q(\omega, |v-u|) \mu^{\frac{1}{2}}(u) [\mu^{\frac{1}{2}}(v') f(u') + \mu^{\frac{1}{2}}(u') f(v')]\, du\, d\omega. \end{aligned} \tag{3.8}$$

The integrand here in K_2 can be written as

$$\begin{aligned} & q\mu^{\frac{1}{2}}(v) \cdot [\mu^{-\frac{1}{2}}(v)][\mu^{\frac{1}{2}}(u)\mu^{\frac{1}{2}}(v') f(u') + \mu^{\frac{1}{2}}(u')\mu^{\frac{1}{2}}(u) f(v')] \\ &= q\mu^{\frac{1}{2}}(v)\mu(u)[\mu^{-\frac{1}{2}}(v)\mu^{-\frac{1}{2}}(u)\mu^{\frac{1}{2}}(v') f(u') + \mu^{-\frac{1}{2}}(v)\mu^{-\frac{1}{2}}(u)\mu^{\frac{1}{2}}(u') f(v')] \\ &= q\mu^{\frac{1}{2}}(v)\mu(u)[\mu^{-\frac{1}{2}}(u') f(u') + \mu^{-\frac{1}{2}}(v') f(v')]. \end{aligned}$$

Therefore

$$K_2 f = \mu^{\frac{1}{2}}(v) \iint q(\omega, |u-v|)\mu(u)[\mu^{-\frac{1}{2}}(u') f(u') + \mu^{-\frac{1}{2}}(v') f(v')]\, d\omega\, du. \tag{3.9}$$

We emphasize one important point: Grad [31], Kawashima [38] and others normalize the Maxwellian μ as

$$\mu = e^{-\frac{1}{2}|v|^2}.$$

Thus the exponents differ by a factor of 2.

3.2. Computation of the Integral Operator

Now write

$$K_i f = \int k_i(u, v) f(u)\, du \qquad (i = 1, 2). \tag{3.10}$$

Clearly,

$$k_1(u, v) = \mu^{\frac{1}{2}}(u)\mu^{\frac{1}{2}}(v) \int_{|\omega|=1} q(\omega, |v-u|)\, d\omega \tag{3.11}$$

and hence k_1 symmetric, but the computation of k_2 is not so simple. We follow Grad's computation [31]. In order to conform to Grad's approach (which has become standard), we change notation slightly to more explicitly indicate the arguments of q. Write

$$V = u - v, \tag{3.12}$$

$$\begin{aligned} u' &= u + a\omega = u + (\omega \cdot (v - u))\omega = u - (\omega \cdot V)\omega, \\ v' &= v - a\omega = v - (\omega \cdot (v - u))\omega = v + (\omega \cdot V)\omega. \end{aligned} \tag{3.13}$$

Take polar coordinates with V as pole:

$$\begin{aligned} &\omega = (\sin\theta\cos\epsilon, \sin\theta\sin\epsilon, \cos\theta) \text{ so that} \\ &\theta = 0 \text{ corresponds to head-on collisions,} \\ &\theta = \frac{\pi}{2} \text{ corresponds to grazing collisions,} \end{aligned} \tag{3.14}$$

and thus

$$\omega \cdot V = |V|\cos\theta,\ 0 < \theta < \frac{\pi}{2},\ 0 \le \epsilon < 2\pi \tag{3.15}$$

and we write collision operator in the form

$$Q(f,f) = \iiint [f(u')f(v') - f(u)f(v)]B(\theta, |V|)\, d\theta\, d\epsilon\, du. \tag{3.16}$$

As special cases we mention the **hard sphere**, with diameter b_0 for which

$$B(\theta, |V|) = b_0|V|\sin\theta\cos\theta$$

and the **inverse power law** for which

$$B(\theta, |V|) = |V|^\gamma \beta(\theta). \tag{3.17}$$

Now write

$$V = (V \cdot \omega)\omega + (V \cdot \omega_\perp)\omega_\perp \qquad (\omega_\perp \perp \omega). \tag{3.18}$$

Consider a rotation $\omega \to \omega_\perp$. Then

$$\begin{aligned} u' &= u - (\omega \cdot V)\omega = u - v + v - (\omega \cdot V)\omega \\ &= v + [V - (\omega \cdot V)\omega] = v + (\omega_\perp \cdot V)\omega_\perp \\ &= v + (\omega \times (V \times \omega)). \end{aligned}$$

Thus under a rotation of ω through $\pi/2$,

$$\theta \to \frac{\pi}{2} - \theta,\ \epsilon \to \epsilon \pm \pi \tag{3.19}$$

we get $u' \to v'$. Similarly $v' \to u'$. Hence such a rotation simply interchanges u', v'. Set

$$B^*(\theta, |V|) = \frac{1}{2}\left[B\Big(\theta, |V|\Big) + B\Big(\frac{\pi}{2} - \theta, |V|\Big)\right]. \tag{3.20}$$

Then

$$K_2 f = 2\mu^{\frac{1}{2}}(v) \iiint \mu^{-\frac{1}{2}}(v') f(v') B^*(\theta, |V|) \mu(u)\, d\theta\, d\epsilon\, du \tag{3.21}$$

because such a rotation has unit Jacobian.

Next, note the map $(u, v) \mapsto (u', v')$ is invariant under $\omega \mapsto -\omega$, i.e.,

$$\begin{aligned} \theta &\to \pi - \theta \\ \varepsilon &\to \epsilon \pm \pi \\ \cos\theta &\to -\cos\theta \\ \sin\theta &\to +\sin\theta \\ \sin\varepsilon &\to -\sin\epsilon \\ \cos\varepsilon &\to -\cos\epsilon. \end{aligned} \tag{3.22}$$

Thus we can extend B^* to $(0, \pi)$ using

$$B^*(\pi - \theta, |V|) = B^*(\theta, |V|). \tag{3.23}$$

Then we can integrate with respect to θ as

$$\frac{1}{2}\int_0^\pi (\cdots)\, d\theta. \tag{3.24}$$

Now ω ranges over a full sphere. Let $\tilde{Q} = \frac{B^*}{|\sin\theta|}$. Then, after changing $u \to V = u - v$, we can write

$$K_2 f = \mu^{\frac{1}{2}}(v) \iiint \mu^{-\frac{1}{2}}(v') \mu(u) f(v') \tilde{Q}(|V|\cos\theta, |V|\sin\theta)\, d\omega\, dV. \tag{3.25}$$

We write the arguments of $\tilde{Q}$ as above for convenience, as will be seen directly. Decompose

$$\begin{aligned} V &= (V \cdot \omega)\omega + (V - (V \cdot \omega)\omega) \\ &= \xi + w \end{aligned} \tag{3.26}$$

where

$$\xi = (V \cdot \omega)\omega, \;\; w = \omega \times (V \times \omega). \tag{3.27}$$

Then

$$\tilde{Q} = \tilde{Q}(|V|\cos\theta, |V|\sin\theta) = \tilde{Q}(|\xi|, |w|). \tag{3.28}$$

In (3.25) we will integrate with respect to w first, then ξ, then ω. One integrates first over the plane w (which is perpendicular to ω). Then we combine the one–dimensional ξ integration in the direction ω with the integral over ω to get a three dimensional integration of $|\xi|\omega \equiv \xi$. There is a factor of 2 because in the three–dimensional integral with respect to ξ we have $-\infty < \xi < \infty$. Thus we have

$$dV = 2\,dw\,d|\xi|. \tag{3.29}$$

For fixed ω the change of variables $V \to (\xi, w)$ is a rotation with unit Jacobian. Hence

$$d\omega\, dV = d\omega \cdot 2\,dw\,d|\xi| = 2\,dw \cdot \frac{|\xi|^2\,d|\xi|\,d\omega}{|\xi|^2} = \frac{2\,d\xi\,dw}{|\xi|^2}. \tag{3.30}$$

Then

$$\begin{aligned} v' &= v + (\omega\cdot V)\omega = v + \xi \\ u &= v + V = v + \xi + w \end{aligned} \tag{3.31}$$

and

$$K_2 f = 2\mu^{\frac{1}{2}}(v)\iiint \mu^{-\frac{1}{2}}(v+\xi)\mu(v+\xi+w)f(v+\xi)\tilde{Q}(|\xi|,|w|)\frac{dw\,d\xi}{|\xi|^2}. \tag{3.32}$$

Let

$$\eta = v + \xi,\ \zeta = \frac{1}{2}(v+\eta). \tag{3.33}$$

Then

$$K_2 f = 2\mu^{\frac{1}{2}}(v)\iiint \mu^{-\frac{1}{2}}(\eta)\mu(\eta+w)f(\eta)\tilde{Q}(|\eta - v|, |w|)\,dw\frac{d\eta}{|\eta - v|^2}. \tag{3.34}$$

Now

$$\mu^{\frac{1}{2}}(v)\mu^{-\frac{1}{2}}(\eta)\mu(\eta+w) \tag{3.35}$$

has in the exponent

$$-\frac{1}{2}|v|^2 + \frac{1}{2}|\eta|^2 - |\eta + w|^2 = -\frac{1}{2}|v|^2 - 2\eta\cdot w - \frac{1}{2}|\eta|^2 - |w|^2. \tag{3.36}$$

But

$$w\cdot\zeta = \frac{1}{2}w\cdot(v+\eta) = \frac{1}{2}w\cdot(2v+\xi) = w\cdot v \tag{3.37}$$

because $w \cdot \xi = 0$ by definition. Hence

$$w \cdot \eta = w \cdot v \quad \text{which implies} \quad w \cdot \zeta = w \cdot \eta. \tag{3.38}$$

Thus this exponent is

$$\begin{aligned} &-\frac{1}{2}|v|^2 - 2w \cdot \zeta - \frac{1}{2}|2\zeta - v|^2 - |w|^2 \\ &= -\frac{1}{2}|v|^2 - 2w \cdot \zeta - 2|\zeta|^2 + 2\zeta \cdot v - |w|^2 - \frac{1}{2}|v|^2 \\ &= -|w + \zeta|^2 - |v|^2 - |\zeta|^2 + 2\zeta \cdot v \\ &= -|w + \zeta|^2 - \frac{1}{4}|v - \eta|^2 \\ &= -|w + \zeta|^2 - \frac{1}{4}\xi^2. \end{aligned} \tag{3.39}$$

Therefore

$$K_2 f = 2 \int e^{-\frac{\xi^2}{4}} e^{-|w+\zeta|^2} f(\eta) \tilde{Q}(|\eta - v|, |w|)\, dw \frac{d\eta}{|\eta - v|^2} \tag{3.40}$$

so that

$$k_2(v, \eta) = \frac{2}{|\eta - v|^2} e^{-\frac{1}{4}|\eta - v|^2} \int e^{-|w+\zeta|^2} \tilde{Q}(|\eta - v|, |w|)\, dw. \tag{3.41}$$

Now resolve

$$\zeta = (\zeta \cdot \hat{u})\hat{u} + \zeta_2 \equiv \zeta_1 + \zeta_2, \ \zeta_1 \cdot \zeta_2 = 0, \ \hat{u} = \frac{\xi}{|\xi|}. \tag{3.42}$$

Then

$$\begin{aligned} \zeta_1^2 &= \left(\frac{\xi}{|\xi|} \cdot \zeta\right)^2 = \left(\frac{\xi}{2|\xi|} \cdot (v + \eta)\right)^2 = \left(\frac{(\eta - v) \cdot (\eta + v)}{2|\eta - v|}\right)^2 \\ &= \frac{1}{4} \frac{(|\eta|^2 - |v|^2)^2}{|\eta - v|^2} \end{aligned} \tag{3.43}$$

and

$$\begin{aligned} |w + \zeta|^2 &= |w + \zeta_1 + \zeta_2|^2 \\ &= |\zeta_1|^2 + |w + \zeta_2|^2 + 2\zeta_1 \cdot (w + \zeta_2). \end{aligned} \tag{3.44}$$

But $\zeta_1 \cdot \zeta_2 = 0$ by definition, and $\zeta_1 \cdot w = 0$ because ζ_1 has direction ξ, and $\xi \cdot w = 0$. Hence

$$k_2(v, \eta) = \frac{2}{|v - \eta|^2} e^{-[\frac{1}{4}|\eta - v|^2 + \frac{1}{4}\frac{(|\eta|^2 - |v|^2)^2}{|\eta - v|^2}]} \int e^{-|w+\zeta_2|^2} \tilde{Q}(|\eta - v|, |w|)\, dw. \tag{3.45}$$

3.2.1. The hard–sphere case. Now let us specialize to the **hard–sphere** case:

$$B(\theta, |V|) = b_0 |V| \sin\theta \cos\theta, \quad \text{so} \tag{3.46}$$

$$\begin{aligned} B\left(\frac{\pi}{2} - \theta, |V|\right) &= b_0 |V| \sin\left(\frac{\pi}{2} - \theta\right) \cos\left(\frac{\pi}{2} - \theta\right) \\ &= b_0 |V| \cos\theta \cdot \sin\theta = B(\theta, |V|). \end{aligned} \tag{3.47}$$

Thus

$$B^*(\theta, |V|) = B(\theta, |V|) \tag{3.48}$$

so that

$$\tilde{Q}(|V|\cos\theta, |V|\sin\theta) = b_0 |V| \cos\theta, \quad \text{i.e.,} \quad \tilde{Q}(|v|, |w|) = b_0 |v|. \tag{3.49}$$

Now using this we compute (recall that S^2_+ was defined in Chapter 2)

$$\begin{aligned} k_1(v, u) &= \mu^{\frac{1}{2}}(v) \int_{|\omega|=1} q(\omega, |v-u|) \mu^{\frac{1}{2}}(u)\, d\omega \\ &= \mu^{\frac{1}{2}}(v) \int_{S^2_+} b_0\, \omega \cdot (v-u) \mu^{\frac{1}{2}}(u)\, d\omega \\ &= b_0 \mu^{\frac{1}{2}}(v) \mu^{\frac{1}{2}}(u) \cdot 2\pi \int_0^{\pi/2} |v-u| \cos\theta \sin\theta\, d\theta \\ &= \pi b_0 |u-v| \exp\left[-\frac{1}{2}(|u|^2 + |v|^2)\right]. \end{aligned} \tag{3.50}$$

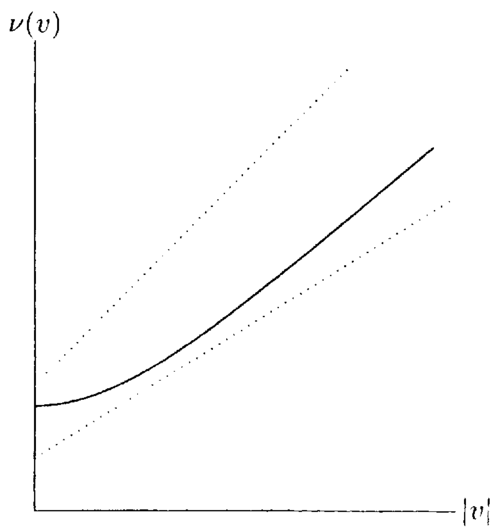

Fig. 3.1 *The Collision Frequency $\nu(v)$ and its linear bounds*

Similarly,

$$
\begin{aligned}
\nu(v) &= \iint_{S_2^+} q(\omega, |v-u|)\mu(u)\,du\,d\omega \\
&= \pi b_0 \int |v-u| e^{-|u|^2}\,du \\
&= \pi b_0 \int |z| e^{-|z+v|^2}\,dz \\
&= \pi b_0 e^{-|v|^2} \int |z| e^{-|z|^2 - 2z\cdot v} dz \\
&= 2\pi^2 b_0 e^{-|v|^2} \int_0^\infty r^3 e^{-r^2} \int_0^\pi e^{-2r|v|\cos\theta} \sin\theta\,d\theta\,dr \\
&= 2\pi^2 b_0 e^{-|v|^2} \int_0^\infty \frac{r^3 e^{-r^2}}{2r|v|} \int_0^\pi \frac{d}{d\theta} e^{-2r|v|\cos\theta}\,d\theta \\
&= \frac{\pi^2 b_0 e^{-|v|^2}}{|v|} \int_0^\infty r^2 e^{-r^2}\left[e^{2r|v|} - e^{-2r|v|}\right] dr \\
&= \frac{\pi^2 b_0}{|v|}\left[\int_0^\infty r^2 e^{-(r-|v|)^2}\,dr - \int_0^\infty r^2 e^{-(r+|v|)^2}\,dr\right] \\
&= \frac{\pi^2 b_0}{|v|}\left[\int_{-|v|}^\infty (z+|v|)^2 e^{-z^2}\,dz - \int_{|v|}^\infty (z-|v|)^2 e^{-z^2}\,dz\right] \\
&= \frac{\pi^2 b_0}{|v|}\left[\int_{-|v|}^{|v|} (z+|v|)^2 e^{-z^2}\,dz + \int_{|v|}^\infty e^{-z^2}\{(z+|v|)^2 - (z-|v|)^2\}\,dz\right] \\
&= \frac{\pi^2 b_0}{|v|}\left[2\int_0^{|v|} (|v|^2+z^2) e^{-z^2} dz + 4\int_{|v|}^\infty |v| z e^{-z^2} dz\right] \\
&= \frac{\pi^2 b_0}{|v|}\left[2|v|^2 \int_0^{|v|} e^{-z^2} dz + 2\left\{\frac{-|v|e^{-|v|^2}}{2} + \frac{1}{2}\int_0^{|v|} e^{-z^2} dz\right\} + 2|v|e^{-|v|^2}\right] \\
&= \pi^2 b_0 \left[\left(2|v| + \frac{1}{|v|}\right)\int_0^{|v|} e^{-z^2} dz + e^{-|v|^2}\right].
\end{aligned}
$$

This is the explicit form of $\nu(v)$ in the hard-sphere case. In order to express $k_2(u,v)$ we need

$$
\begin{aligned}
(3.51) \qquad \int_{\mathbb{R}^2} e^{-|w+\zeta_2|^2} \tilde{Q}(|\eta - v|, |w|)\,dw &= b_0 |\eta - v| \int_{\mathbb{R}^2} e^{-|w+\zeta_2|^2}\,dw \\
&= |\eta - v| \cdot \pi b_0.
\end{aligned}
$$

Therefore

$$
(3.52) \qquad k_2(u,v) = \frac{2\pi b_0}{|u-v|} \exp\left[-\frac{1}{4}|u-v|^2 - \frac{1}{4}\frac{(|u|^2 - |v|^2)^2}{|u-v|^2}\right].
$$

and thus k_2 is **symmetric**. Notice that neither k_1 nor k_2 is a convolution kernel.

This calculation was first done by Hilbert in 1912.

3.3. Estimates on the Integral Operator

3.3.1. Bounds on the Collision Frequency $\nu(v)$. From above we have

$$\nu(v) = c \int |v - u| \mu(u) \, du \tag{3.53}$$

for some $c > 0$ and $\mu(u) = e^{-|u|^2}$.

a) We have the pointwise bounds (see Fig. 3.1)

$$0 < \nu_0 \leq \nu_1(1 + |v|) \leq \nu(v) \leq \nu_2(1 + |v|) \tag{3.54}$$

for all v. Indeed, we certainly have

$$\nu(v) \leq c \int (|v| + |u|) \mu(u) \, du \leq \nu_2(1 + |v|) \tag{3.55}$$

everywhere.

For $|v| \geq 1$,

$$\begin{aligned} \nu(v) &\geq c\chi_{|v| \geq 1} \int \big||v| - |u|\big| \mu(u) \, du \\ &\geq c\chi_{|v| \geq 1} \int_{|u| < 1/2} \big||v| - |u|\big| \mu(u) \, du \\ &\geq c\chi_{|v| \geq 1} \int_{|u| < 1/2} \left(|v| - \frac{1}{2}\right) \mu(u) \, du \\ &\geq c|v|\chi_{|v| \geq 1}. \end{aligned} \tag{3.56}$$

In the set $|v| \leq 1$ we can write

$$\nu(v) \geq c \int_{|u| > 2} \big||u| - |v|\big| \mu(u) \, du \geq c \int_{|u| > 2} \mu(u) \, du > 0. \tag{3.57}$$

b) (Monotonicity in the radial direction)

We have

$$\frac{\partial}{\partial |v|} \nu(v) > 0. \tag{3.58}$$

For, from definition

$$\frac{\nu(v)}{\pi b_0} = \int |v - u| \mu(u) \, du. \tag{3.59}$$

Thus

$$\frac{\partial}{\partial v_i}\frac{\nu(v)}{\pi b_0} = \int \frac{v_i - u_i}{|v-u|}\mu(u)\,du \tag{3.60}$$

so that

$$\begin{aligned}
\frac{1}{\pi b_0}\frac{\partial \nu(v)}{\partial |v|} &= \sum_i \frac{v_i}{|v|}\frac{\partial}{\partial v_i}\nu(v)\cdot\frac{1}{\pi b_0} \\
&= \frac{1}{|v|}\int \frac{v\cdot(v-u)}{|v-u|}\mu(u)\,du \\
&= \frac{1}{|v|}\int \frac{v\cdot z}{|z|}e^{-|v-z|^2}\,dz \\
&= \frac{1}{|v|}\int_0^\infty 2\pi \int_0^\pi |z|^2\frac{|v||z|\cos\theta}{|z|}e^{-|v|^2-|z|^2+2|v||z|\cos\theta}\sin\theta\,d\theta\,d|z|.
\end{aligned}$$

Now denote by A the angular integral appearing here, and write $a = 2|v||z|$:

$$\begin{aligned}
A &\equiv \int_0^\pi e^{2|v||z|\cos\theta}\sin\theta\cos\theta\,d\theta \\
&= \int_0^{\pi/2} e^{a\cos\theta}\sin\theta\cos\theta\,d\theta + \int_{\pi/2}^{\pi} e^{a\cos\theta}\sin\theta\cos\theta\,d\theta.
\end{aligned} \tag{3.61}$$

In the second integral let $\phi = \theta - \frac{\pi}{2}$. Then we get

$$A = \int_0^{\pi/2}(e^{a\cos\theta} - e^{-a\sin\theta})\cos\theta\sin\theta\,d\theta > 0. \tag{3.62}$$

3.3.2. Bounds on the Kernel.

We know that

$$k_1(u,v) = c|u-v|e^{-\frac{1}{2}(|u|^2+|v|^2)}. \tag{3.63}$$

Thus

$$\begin{aligned}
\int k_1(u,v)\,du &\le c\int(|u|+|v|)e^{-\frac{1}{2}(|u|^2+|v|^2)}\,du \\
&\le c(1+|v|)e^{-\frac{1}{2}|v|^2} \\
&\le ce^{-\frac{1}{4}|v|^2}.
\end{aligned} \tag{3.64}$$

Similarly

$$\begin{aligned}
\int k_1^2(u,v)\,du &\le c\int(|u|+|v|)^2e^{-(|u|^2+|v|^2)}\,du \\
&\le c(1+|v|^2)e^{-|v|^2} \\
&\le ce^{-\frac{1}{4}|v|^2}.
\end{aligned} \tag{3.65}$$

Hence

$$\sup_v \int k_1(u,v)\,du < \infty, \quad \sup_v \int k_1^2(u,v)\,du < \infty. \tag{3.66}$$

Clearly we have

$$k_2(u,v) \le \frac{c}{|u-v|} e^{-\frac{1}{4}|u-v|^2} \tag{3.67}$$

and hence

$$\sup_v \int k_2(u,v)\,du < \infty, \quad \sup_v \int k_2^2(u,v)\,du < \infty. \tag{3.68}$$

LEMMA 3.3.1 *Let $\alpha \ge 0$; then for $k = k_1 + k_2$,*

$$\int |k(u,v)|(1+u^2)^{-\alpha/2}\,du \le c(1+|v|^2)^{-\frac{1}{2}(\alpha+1)}. \tag{3.69}$$

Proof. This is easy for k_1:

(3.70)

$$\begin{aligned}
&\int |k_1(u,v)|(1+|u|^2)^{-\alpha/2}\,du \\
&\le c\int |u-v| e^{-\frac{1}{2}(|u|^2+|v|^2)}(1+|u|^2)^{-\alpha/2}\,du \\
&= \int_{\{|u|<|v|\}} + \int_{\{|u|>|v|\}} \\
&\le c|v|e^{-\frac{1}{2}|v|^2}\int e^{-\frac{1}{2}|u|^2}(1+|u|^2)^{-\alpha/2}\,du + c(1+|v|^2)^{-\alpha/2}e^{-\frac{1}{2}|v|^2}\int |u|e^{-\frac{1}{2}|u|^2}\,du \\
&\le ce^{-\frac{1}{4}|v|^2}
\end{aligned}$$

which is more than enough for large $|v|$. The small $|v|$ case is addressed below. For k_2 we have

$$\begin{aligned}
\int |k_2(u,v)|(1+|u|^2)^{-\alpha/2}\,du &\le c\int \frac{e^{-\frac{1}{4}\left(|u-v|^2+\frac{(|u|^2-|v|^2)^2}{|u-v|^2}\right)}}{|u-v|(1+|u|^2)^{\alpha/2}}\,du \\
&= \int_{\{|u-v|>\frac{|v|}{2}\}} + \int_{\{|u-v|<\frac{|v|}{2}\}} \\
&\equiv I_1 + I_2.
\end{aligned} \tag{3.71}$$

Now

$$I_1 \le \int_{|u-v|>\frac{|v|}{2}} \frac{e^{-\frac{1}{8}|u-v|^2}\cdot e^{-\frac{1}{8}\cdot\frac{|v|^2}{4}}}{|v|/2}\,du \le c|v|^{-1}e^{-\frac{1}{32}|v|^2} \tag{3.72}$$

which is more than enough. The estimate for small $|v|$ is done below.

For I_2 we write

$$|u|^2 - |v|^2 = |u-v|^2 + 2v \cdot (u-v). \tag{3.73}$$

Set

$$r = |u-v|, \ \ v \cdot (u-v) = r|v| \cos\theta. \tag{3.74}$$

Thus

$$(|u|^2 - |v|^2)^2 = (r^2 + 2r|v|\cos\theta)^2. \tag{3.75}$$

On the set $|u-v| < \frac{|v|}{2}$ we have

$$|u| \geq |v| - |u-v| > |v|/2. \tag{3.76}$$

Therefore

$$I_2 \leq c(1+|v|^2)^{-\alpha/2} \int_0^\infty re^{-r^2/4} \cdot 2\pi \int_0^\pi e^{-\frac{1}{4}(r+2|v|\cos\theta)^2} \sin\theta \, d\theta \, dr. \tag{3.77}$$

Change variables by

$$w = \frac{1}{2}(r + 2|v|\cos\theta).$$

Then

$$\int_0^\pi e^{-\frac{1}{4}(r+2|v|\cos\theta)^2} \sin\theta \, d\theta = |v|^{-1} \int_{\frac{r-2|v|}{2}}^{\frac{r+2|v|}{2}} e^{-w^2} \, dw \leq c|v|^{-1}. \tag{3.78}$$

Hence

$$I_2 \leq c(1+|v|^2)^{-\alpha/2} \cdot |v|^{-1}. \tag{3.79}$$

This proves the Lemma for large $|v|$.

For the small $|v|$ case, let $|v| \leq 1$, say. Then

$$\begin{aligned} \int |k(u,v)|(1+u^2)^{-\alpha/2} \, du &\leq \sup_v \int |k(u,v)| \, du \\ &\leq c \\ &= \frac{c}{(1+|v|^2)^{\frac{\alpha+1}{2}}} \cdot (1+|v|^2)^{\frac{\alpha+1}{2}} \\ &\leq \frac{c}{(1+|v|^2)^{\frac{\alpha+1}{2}}} \end{aligned} \tag{3.80}$$

as desired.

3.4. Properties of L

Recall that

$$L[f] = \nu(v)f - Kf \tag{3.81}$$

where

$$\begin{aligned} K &= K_2 - K_1, \\ K_i f &= \int k_i(u,v)f(u)\,du \qquad (i=1,2). \end{aligned} \tag{3.82}$$

Also, the equation for the perturbation f is

$$f_t + v\cdot\nabla_x f + L[f] = \mu^{-1/2}(v)Q^*\left(\mu^{\frac12}f, \mu^{\frac12}f\right) \tag{3.83}$$

where

$$L[f] = -2Q^*\left(\mu, \mu^{\frac12}f\right)\cdot\mu^{-1/2}. \tag{3.84}$$

Recall from Chapter 1 the identity

$$\begin{aligned} &\int Q^*(F,G)\phi(v)\,dv \\ &= \frac12\iiint q[F(v')G(u') + F(u')G(v') - F(u)G(v) - F(v)G(u)]\phi(v)\,d\omega\,du\,dv \\ &= \frac12\iiint q[F(v')G(u') + F(u')G(v') - F(u)G(v) - F(v)G(u)]\phi(u)\,d\omega\,du\,dv \\ &= \frac12\iiint q[F(v')G(u') + F(u')G(v') - F(u)G(v) - F(v)G(u)](-\phi(v'))\,d\omega\,du\,dv \\ &= \frac12\iiint q[F(v')G(u') + F(u')G(v') - F(u)G(v) - F(v)G(u)](-\phi(u'))\,d\omega\,du\,dv. \end{aligned} \tag{3.85}$$

Symmetry is an easy property because we already know that $k_i(u,v) = k_i(v,u)$: hence

$$\int f\cdot L[g]\,dv = \int g\cdot L[f]\,dv \tag{3.86}$$

provided

$$f, g \in D(L) = \{f\in L^2:\ L[f]\in L^2\}. \tag{3.87}$$

Next, assuming $q\ge 0$, we claim that L is **non–negative**: by (3.84), this is equivalent to

$$\int \mu^{-\frac12} f Q^*(\mu, \mu^{\frac12}f)\,dv \le 0. \tag{3.88}$$

Add the four choices in identity (3.85) with $\phi = \mu^{-\frac{1}{2}} f$ to get

$$\begin{aligned}
&\int 2\mu^{-\frac{1}{2}}(v) f(v) Q^*(\mu, \mu^{\frac{1}{2}} f)\, dv \\
&\equiv \iiint q[\mu(v')\mu^{\frac{1}{2}}(u') f(u') + \mu(u')\mu^{\frac{1}{2}}(v') f(v') - \mu(v)\mu^{\frac{1}{2}}(u) f(u) - \mu(u)\mu^{\frac{1}{2}}(v) f(v)]\cdot \\
&\qquad [\mu^{-\frac{1}{2}}(v) f(v) + \mu^{-\frac{1}{2}}(u) f(u) - \mu^{-\frac{1}{2}}(v') f(v') - \mu^{-\frac{1}{2}}(u') f(u')]\, d\omega\, du\, dv \\
&= -\iiint q\mu(v')\mu(u')[\mu^{-\frac{1}{2}}(u') f(u') + \mu^{-\frac{1}{2}}(v') f(v') - \mu^{-\frac{1}{2}}(u) f(u) - \mu^{-\frac{1}{2}}(v) f(v)]\cdot \\
&\qquad [\mu^{-\frac{1}{2}}(v') f(v') + \mu^{-\frac{1}{2}}(u') f(u') - \mu^{-\frac{1}{2}}(v) f(v) - \mu^{-\frac{1}{2}}(u) f(u)]\, d\omega\, du\, dv \\
&= -\iiint q\mu(u)\mu(v)[\mu^{-\frac{1}{2}}(v') f(v') + \mu^{-\frac{1}{2}}(u') f(u') - \mu^{-\frac{1}{2}}(v) f(v) - \mu^{-\frac{1}{2}}(u) f(u)]^2\, d\omega\, du\, dv \\
&\leq 0.
\end{aligned}$$

Of course we can write

$$\begin{aligned}
-L[f] &= 2Q^*(\mu, \mu^{\frac{1}{2}} f)\cdot \mu^{-\frac{1}{2}} \\
&= \mu^{-\frac{1}{2}} \iint q\Big[\mu(v')\mu^{\frac{1}{2}}(u') f(u') + \mu(u')\mu^{\frac{1}{2}}(v') f(v') \\
&\qquad - \mu(v)\mu^{\frac{1}{2}}(u) f(u) - \mu(u)\mu^{\frac{1}{2}}(v) f(v)\Big]\, d\omega\, du \\
&= \iint q\frac{\mu(v')\mu(u')}{\sqrt{\mu(v)}}\Big[\mu^{-\frac{1}{2}}(u') f(u') + \mu^{-\frac{1}{2}}(v') f(v') \\
&\qquad - \mu^{-\frac{1}{2}}(u) f(u) - \mu^{-\frac{1}{2}}(v) f(v)\Big]\, d\omega\, du.
\end{aligned}$$

Suppose that, for $i = 1, 2, 3$,

$$\mu^{-\frac{1}{2}}(u) f(u) = \begin{bmatrix} 1 \\ u_i \\ |u|^2 \end{bmatrix} \quad \text{(a 5-vector)} \tag{3.89}$$

i.e.,

$$f(u) = \mu^{\frac{1}{2}}(u) \begin{bmatrix} 1 \\ u_i \\ |u|^2 \end{bmatrix}. \tag{3.90}$$

Then $L[f] = 0$. In fact, $\int f L[f]\, dv = 0$ if and only if f is a linear combination of these quantities, i.e., if and only if

$$f(u) = \mu^{\frac{1}{2}}(u)(a + b\cdot u + c|u|^2). \tag{3.91}$$

For, we have seen above that

$$\int f L[f]\, dv \geq 0 \tag{3.92}$$

for all f, and that the integrand in this expression is of one sign. Hence

$$\int fL[f]\,dv = 0 \quad \text{if and only if} \tag{3.93}$$
$$\mu^{-\frac{1}{2}}(v')f(v') + \mu^{-\frac{1}{2}}(u')f(u') = \mu^{-\frac{1}{2}}(u)f(u) + \mu^{-\frac{1}{2}}(v)f(v) \quad \text{a.e.}$$

Set

$$g(u) = \mu^{-\frac{1}{2}}(u)f(u); \tag{3.94}$$

then this says that g is a **collision invariant**:

$$g(u') + g(v') = g(u) + g(v). \tag{3.95}$$

From earlier considerations, f must be as given in (3.90).

Hence the eigenvalue $\lambda = 0$ of L has a 5-dimensional null space spanned by

$$\mu^{\frac{1}{2}}(v)\begin{bmatrix} 1 \\ v_i \\ |v|^2 \end{bmatrix}. \tag{3.96}$$

If there are any other discrete eigenvalues, they must be positive since

$$\int fL[f]\,dv \geq 0. \tag{3.97}$$

for all $f \in D(L)$. This follows from the spectral theorem as applied to the self-adjoint operator L.

Denote by $K^{(n)}$ the n'th iterated integral operator, with $K^{(1)} = K$. Carleman [11] showed that $K^{(2)}$ is compact for the hard sphere model. Grad [31] showed that $K^{(3)}$ is square-integrable for more general q, and hence K is compact. Now the spectrum of the self-adjoint operator $f \mapsto \nu(v)f$ is continuous and coincides with the range of $\nu(v)$; thus it is $[\nu_0, \infty)$. By Weyl's Theorem, the essential spectrum of $\nu(v) - K$ is the same as that of $\nu(v)$. (The essential spectrum is the set of accumulation points of the spectrum; equivalently, it is that set which remains after excluding from the spectrum all isolated points which are eigenvalues of finite multiplicity.) See [37] (p. 244) for a statement of a generalized Weyl's theorem. Therefore 0 is an isolated point eigenvalue of L.

LEMMA 3.4.1 *Assume*

$$\int \mu^{+\frac{1}{2}} f\,dv = \int \mu^{+\frac{1}{2}} v_i f\,dv = \int \mu^{+\frac{1}{2}} |v|^2 f\,dv = 0. \tag{3.98}$$

Then there is a $\mu > 0$ *such that*

$$\int fL[f]\,dv \geq \mu \int f^2\,dv. \tag{3.99}$$

If there are any discrete eigenvalues between 0 and ν_0 (to which correspond the eigenfunctions above) we take μ = smallest of them. Otherwise, we take $\mu = \nu_0$.

3.5. Compactness of K

Write $k = |k_1| + |k_2|$. First we note that K is bounded on L^2 via

$$\begin{aligned}|Kf(v)| &\leq \int k^{1/2}(u,v) \cdot k^{1/2}(u,v) f(u)\, du \\ &\leq \left(\int k(u,v)\, du\right)^{1/2} \left(\int k(u,v) f^2(u)\, du\right)^{1/2}\end{aligned} \tag{3.100}$$

Therefore

$$\begin{aligned}\|Kf\|_2^2 &\leq \left[\sup_v \int k(u,v)\, du\right] \cdot \int\int k(u,v) f^2(u)\, du\, dv \\ &\leq \left[\sup_v \int k(u,v)\, du\right] \left[\sup_u \int k(u,v)\, dv\right] \|f\|^2.\end{aligned} \tag{3.101}$$

Consider an integral operator

$$Tf(v) = \int \kappa(u,v) f(u)\, du, \quad T_n f(v) = \int \kappa_n(u,v) f(u)\, du. \tag{3.102}$$

Let $\Omega_n = A_n \cap B_n$ where

$$A_n = \{(u,v) : |u-v| \geq \frac{1}{n}\}; \quad B_n = \{(u,v) : |v| \leq n\}. \tag{3.103}$$

Let $\chi_{\Omega_n}\kappa = \kappa_n$, a cut-off kernel.

LEMMA 3.5.1 (DRANGE [21]) *Assume that $\kappa(u,v) \geq 0$. T is compact on L^2 if*

(i) $\int \kappa(u,v)\, dv$ *is bounded in* u;

(ii) $\kappa \in L^2(\Omega_n)$ *for all* n;

(iii) $\sup_v \int (\kappa - \kappa_n)\, du \to 0$, $n \to \infty$.

Proof. By *(ii)*, T_n is compact on L^2. Now we claim that *(i)* and *(iii)* imply that $\|T_n - T\| \to 0$. Assume that the claim is true. Then T, as a uniform limit of compact operators, is itself compact.

Now

$$\begin{aligned}\|(T-T_n)f\|^2 &= \left\|\int (\kappa(u,v) - \kappa_n(u,v)) f(u)\, du\right\|^2 \\ &= \int \left[\int (\kappa - \kappa_n) f(u)\, du\right]^2 dv\end{aligned}$$

$$\leq \int \left[\left(\int (\kappa - \kappa_n)\, du \right)^{1/2} \left(\int (\kappa - \kappa_n) f^2(u)\, du \right)^{1/2} \right]^2 dv$$
$$= \int \left(\int (\kappa - \kappa_n)(u,v)\, du \right) \left(\int (\kappa - \kappa_n) f^2(u)\, du \right) dv$$
$$\leq \sup_v \int (\kappa - \kappa_n)\, du \cdot \int f^2(u) \left\{ \int (\kappa - \kappa_n)\, dv \right\} du \qquad \text{(Fubini)}$$
$$\leq \sup_v \int (\kappa - \kappa_n)\, du \cdot \sup_u \int \kappa\, dv \cdot \|f\|_{L^2}^2 \qquad (\text{because } \kappa_n \leq \kappa)$$
$$\leq o(1)\|f\|^2.$$

This proves the claim and the Lemma.

Now recall that k_1 is easily L^2 in (u, v), and that

$$k_2(u,v) = \frac{c}{|u-v|} e^{-\frac{1}{4}\left[|u-v|^2 + \frac{(|u|^2 - |v|^2)^2}{|u-v|^2}\right]}. \tag{3.104}$$

Then *(i)* and *(ii)* above clearly hold. Write $k = k_2$ and

$$\begin{aligned} \int (k - k_n)\, du &= \int (k - \chi_{\Omega_n} k)\, du = \int_{\Omega_n^c} k\, du \\ &= \int_{(A_n \cap B_n)^c} k\, du = \int_{A_n^c \cup B_n^c} k\, du \\ &= \int_{\{(u,v):|u-v|<\frac{1}{n}\} \cup \{(u,v):|v|>n\}} k\, du. \end{aligned} \tag{3.105}$$

Thus

$$\sup_v \int (k - k_n)\, du \leq \sup_v \int_{\{u:|u-v|<\frac{1}{n}\}} k\, du + \sup_{\substack{v \\ |v|>n}} \int k\, du. \tag{3.106}$$

Now

$$\int_{|u-v|<\frac{1}{n}} k\, du \leq c \int_{|u-v|<\frac{1}{n}} \frac{e^{-\frac{1}{4}|u-v|^2}\, du}{|u-v|} \leq c \int_0^{\frac{1}{n}} r\, dr \to 0 \tag{3.107}$$

and

$$\sup_{|v|>n} \int k\, du \leq \frac{c}{n} \to 0 \tag{3.108}$$

because we showed in Lemma 3.3.1 that

$$\int (1 + |u|^2)^{-\alpha/2} |k(u,v)|\, du \leq c(1 + |v|^2)^{-(1+\alpha)/2} \quad \text{for all} \quad \alpha \geq 0. \tag{3.109}$$

3.6. Solution Spaces

Write $\langle\xi\rangle = \sqrt{1+|\xi|^2}$. We define for $\ell = 0, 1, \dots$ and $\beta \geq 0$

$$H^\ell(\mathbb{R}^3) = \left\{ u \in L^2(\mathbb{R}^3) : \|u\|_\ell \equiv \left(\int_{\mathbb{R}^3} \langle\xi\rangle^{2\ell} |\hat{u}(\xi)|^2 d\xi \right)^{1/2} < \infty \right\};$$

$$L^p_\beta(\mathbb{R}^3) = \{u \in L^p(\mathbb{R}^3) : \langle x\rangle^\beta u \in L^p(\mathbb{R}^3)\}, \quad \text{with norm} \quad |u|_{p,\beta} = \|\langle x\rangle^\beta u\|_{L^p}.$$

Here $\hat{u}$ denotes the Fourier Transform of u. Note that $L^\infty_\beta \subset L^2_{\beta'}$ for $\beta > \beta' + \frac{3}{2}$. Indeed, assume that $u \in L^\infty_\beta$, i.e.,

$$\langle x\rangle^\beta u \in L^\infty, \quad \beta > \beta' + \frac{3}{2}. \tag{3.110}$$

Then

$$\begin{aligned} |u|^2_{2,\beta'} &\equiv \int \langle x\rangle^{2\beta'} u^2 \, dx = \int \langle x\rangle^{2\beta} u^2 \cdot \langle x\rangle^{2(\beta'-\beta)} \, dx \\ &\leq \sup_x (\langle x\rangle^\beta |u|)^2 \cdot \int \frac{dx}{\langle x\rangle^{2(\beta-\beta')}} \leq c|u|^2_{\infty,\beta}. \end{aligned} \tag{3.111}$$

In particular,

$$\|u\|_{L^2} \leq c|u|_{\infty,\beta}, \quad \beta > \frac{3}{2}. \tag{3.112}$$

Define further

$$\dot{L}^\infty_\beta(\mathbb{R}^3) = \{u \in L^\infty_\beta(\mathbb{R}^3) : \lim_{R\to\infty} |(1-\chi_R)u|_{\infty,\beta} = 0\} \tag{3.113}$$

where χ_R is the characteristic function of $\{|x| \leq R\}$. The norm is the same.

Spaces in x, v are similarly defined:

$$B^\ell_\beta = L^\infty_\beta(v; H^\ell(x)), \quad \text{with norm} \quad \|\!|f|\!\|_{\ell,\beta} = \sup_v \langle v\rangle^\beta \|f(\cdot, v)\|_{H^\ell}. \tag{3.114}$$

Similarly,

$$\dot{B}^\ell_\beta = \dot{L}^\infty_\beta(v; H^\ell(x)) \tag{3.115}$$

with the same norm. Note that all norms are taken first in x, then in v.

In the previous estimates (3.66), (3.68) we have shown that

$$\begin{aligned} |Kf|_{\infty,0} &= \sup_v |Kf(v)| = \sup_v \int |k(u,v)||f(u)| \, du \\ &\leq \sup_v \left(\int k^2(u,v) \, du \right)^{1/2} \|f\|_{L^2} \leq c\|f\|_{L^2}. \end{aligned} \tag{3.116}$$

Moreover, recall from Lemma 3.3.1 that

$$\int |k(u,v)|\langle u\rangle^{-\alpha}\,du \le c\langle v\rangle^{-(\alpha+1)} \quad (\alpha \ge 0). \tag{3.117}$$

Therefore for $\beta \ge 1$,

$$\begin{aligned}
|Kf|_{\infty,\beta} &= \sup_v \langle v\rangle^{\beta}|Kf(v)| \\
&= \sup_v \langle v\rangle^{\beta}\int |k(u,v)||f(u)|\,du \\
&= \sup_v \langle v\rangle^{\beta}\int |k(u,v)|\langle u\rangle^{\beta-1}|f(u)|\langle u\rangle^{-(\beta-1)}\,du \\
&\le \sup_v \langle v\rangle^{\beta}\left[\sup_u \langle u\rangle^{\beta-1}|f(u)|\cdot\int |k(u,v)|\langle u\rangle^{-(\beta-1)}\,du\right] \\
&\le c\sup_v \langle v\rangle^{\beta}\cdot |f|_{\infty,\beta-1}\cdot\langle v\rangle^{-\beta} \\
&\le c|f|_{\infty,\beta-1}.
\end{aligned} \tag{3.118}$$

3.6.1. The "dot" spaces. Why does one need the "dot" spaces? These are required to show that certain semigroups are strongly continuous. Consider for instance the space $\dot{B}^0_\beta = \dot{L}^\infty_\beta(v, L^2(x))$. For $t \ge 0$ consider the semigroup $U(t)$ defined by

$$(U(t)f)(x,v) = f(x-tv,v), \quad f \in \dot{B}^0_\beta.$$

We claim that $t \mapsto U(t)f$ is strongly continuous from $\mathbb{R}^+$ into $\dot{B}^0_\beta$. Indeed $U(t)$ is bounded because

$$|||U(t)f|||_{0,\beta} = \sup_v < v >^{\beta} \|f(\cdot - tv, v)\|_{L^2_x} = |||f|||_{0,\beta}.$$

In order to show the strong continuity, we consider $t \to 0^+$. For any $R > 0$ we have

$$\begin{aligned}
|||U(t)f - f|||_{0,\beta} &= \sup_v < v >^{\beta} \|f(\cdot - tv, v) - f(\cdot, v)\|_{L^2_x} \\
&\le \sup_{|v|>R} < v >^{\beta} \|f(\cdot - tv, v) - f(\cdot, v)\|_{L^2_x} \\
&\quad + \sup_{|v|<R} < v >^{\beta} \|f(\cdot - tv, v) - f(\cdot, v)\|_{L^2_x} \\
&\equiv U_1 + U_2.
\end{aligned}$$

For U_1 we have the estimate

$$U_1 \le 2\sup_{|v|>R} < v >^{\beta} \|f(\cdot, v)\|_{L^2_x}$$

and thus $U_1 \to 0$ as $R \to \infty$. For any large but fixed choice of R, $U_2 \to 0$ as $t \to 0^+$ by integration theory (since v is bounded in the expression U_2). Therefore $U(t)$ is strongly continuous as claimed. By the Hille–Yosida theorem, the infinitesimal generator A of $U(t)$ has dense domain $D(A)$ and

$$(Af)(x,v) = -v \cdot \nabla_x f \quad \text{for} \quad f \in D(A).$$

3.7. An Orthonormal Basis for N(L)

We know $N(L)$ is the linear span of $\{\mu^{1/2}\psi_1, \dots, \mu^{1/2}\psi_5\}$ where

$$\psi_1 = 1;\ \psi_{j+1} = v_j,\ j = 1,2,3;\ \psi_5 = |v|^2. \tag{3.119}$$

What follows is a straightforward calculus computation. A relevant integral is

$$\int_0^\infty x^{2n} e^{-ax^2}\, dx = \frac{1 \cdot 3 \cdot 5 \cdots (2n-1)}{2^{n+1} a^n} \sqrt{\frac{\pi}{a}} \quad (n = 0, 1, \dots)\,. \tag{3.120}$$

We begin with

$$\int \mu\, dv = \int e^{-|v|^2}\, dv = \pi^{3/2}. \tag{3.121}$$

Set

$$e_1 = \frac{\mu^{1/2}}{\pi^{3/4}}. \tag{3.122}$$

Then $\|e_1\|_{L^2} = 1$. Next, compute

$$\begin{aligned} \int v_1^2 \mu\, dv &= \int_0^\infty r^4 e^{-r^2}\, dr \int_0^{2\pi}\int_0^\pi \cos^2\theta \sin^2\phi \sin\phi\, d\theta\, d\phi \\ &= \int_0^\infty r^4 e^{-r^2} dr \int_0^{2\pi} \cos^2\theta\, d\theta \int_0^\pi \sin^3\phi\, d\phi \\ &= \int_0^\infty r^4 e^{-r^2} dr \cdot \pi \cdot \frac{4}{3} = \frac{4\pi}{3} \cdot \frac{3}{2^3} \cdot \sqrt{\pi} = \frac{\pi^{3/2}}{2}. \end{aligned} \tag{3.123}$$

Thus we set

$$e_{j+1} = \frac{\sqrt{2}}{\pi^{3/4}} v_j \mu^{1/2} \quad (j = 1,2,3). \tag{3.124}$$

Then $\|e_{j+1}\|_{L^2} = 1$ and $(e_j, e_1) = 0,\ j = 1,2,3.$

For e_5 we compute

$$\int |v|^4 \mu\, dv = 4\pi \int_0^\infty r^6 e^{-r^2} dr = 4\pi \cdot \frac{3 \cdot 5}{2^4} \cdot \sqrt{\pi} = \frac{15\pi^{3/2}}{4}. \tag{3.125}$$

Now we consider

$$e_5 = \alpha(|v|^2 - \beta)\mu^{1/2}. \tag{3.126}$$

We want $(e_5, e_1) = 0$ and $\|e_5\|_{L^2} = 1$. Now

$$\begin{aligned}(e_5, e_1) &= \alpha \int |v|^2 \mu \, dv - \alpha\beta \int \mu \, dv \\ &= 4\pi\alpha \int_0^\infty r^4 e^{-r^2} dr - \alpha\beta \cdot \pi^{3/2} \\ &= 4\pi\alpha \cdot \frac{3}{8}\sqrt{\pi} - \alpha\beta\pi^{3/2} \\ &= \pi^{3/2}\alpha\left(\frac{3}{2} - \beta\right).\end{aligned} \tag{3.127}$$

Therefore we take $\beta = 3/2$. Then $(e_5, e_1) = 0$ and

$$\begin{aligned}\|e_5\|_{L^2}^2 &= \alpha^2 \int (|v|^2 - \beta)^2 \mu \, dv = \alpha^2 \int (|v|^4 - 2\beta|v|^2 + \beta^2)\mu \, dv \\ &= \alpha^2 \left[\frac{15}{4}\pi^{3/2} - 2\beta \cdot \frac{3}{2}\pi^{3/2} + \beta^2 \cdot \pi^{3/2}\right] \\ &= \alpha^2\pi^{3/2}\left[\frac{15}{4} - \frac{9}{2} + \frac{9}{4}\right] = \alpha^2\pi^{3/2} \cdot \frac{6}{4} = \frac{3}{2}\alpha^2\pi^{3/2}.\end{aligned} \tag{3.128}$$

Thus we take

$$\alpha = \sqrt{\frac{2}{3}}\,\pi^{-3/4}; \tag{3.129}$$

then

$$\|e_5\|_{L^2} = 1. \tag{3.130}$$

Therefore we take

$$e_5 = \sqrt{\frac{2}{3}}\,\pi^{-3/4}\left(|v|^2 - \frac{3}{2}\right)\mu^{1/2} \tag{3.131}$$

and an orthonormal basis for $N(L)$ is $\{e_1, \ldots, e_5\}$ as claimed. Let P_0 be the orthogonal projection from $L^2(v)$ onto $N(L)$:

$$P_0 f = \sum_{k=1}^{5} (f, e_k)e_k. \tag{3.132}$$

From Lemma 3.4.1 we know that there exist $\delta_1 > 0$ such that

$$(f, L[f]) \geq \delta_1 \|(I - P_0)f\|_2^2, \ f \in D(L). \tag{3.133}$$

LEMMA 3.7.1 *(3.133) holds for* $f \in L^2_\beta(v)$, $\beta \geq 1$.

Proof. This will follow from the inclusion $L^2_\beta \subset D(L)$. Recall that $L[f] = \nu(v)f - Kf$. We will show that $\nu(v)f \in L^2$, $Kf(v) \in L^2$ for $f \in L^2_\beta$.

Since $\nu(v) \leq c(1+|v|) \leq c\langle v\rangle$ and $\beta \geq 1$,

$$\begin{aligned}\int \nu^2(v) f^2(v)\,dv &\leq c \int \langle v\rangle^2 f^2(v)\,dv \\ &\leq c \int \langle v\rangle^{2\beta} f^2(v)\,dv \\ &= c|f|^2_{2,\beta} < \infty. \end{aligned} \tag{3.134}$$

Let $k = |k_1| + |k_2|$. Previously, when showing that K is bounded on L^2 in (3.100), we showed that

$$|Kf(v)| \leq \left(\int k\,du\right)^{1/2} \left(\int kf^2(u)\,du\right)^{1/2} \tag{3.135}$$

Hence by (3.66), (3.68),

$$\begin{aligned}\|Kf\|_2^2 &\leq c\left[\sup_v \int k\,du\right] \int f^2(u) \left[\int k\,dv\right] du \\ &\leq c\|f\|_2^2 \cdot \left(\sup_v \int k\,du\right) \cdot \left(\sup_u \int k\,dv\right) \\ &\leq c\|f\|_2^2 < \infty \end{aligned} \tag{3.136}$$

since $L^2_\beta \subset L^2$ for all $\beta \geq 0$.

3.8. Estimates on the Nonlinear Term

We are always assuming the hard-sphere case. Recall that the equation for the perturbation f from equilibrium is

$$f_t + v \cdot \nabla_x f + L[f] = \Gamma(f, f).$$

We define the symmetric form

$$\Gamma(f,g) \equiv \mu^{-\frac{1}{2}} Q^*(\mu^{\frac{1}{2}} f, \mu^{\frac{1}{2}} g). \tag{3.137}$$

We begin with some properties of $\Gamma(f,g)$:

(i) $P_0\Gamma(f,g) = 0$,

(ii) $|\nu^{-1}(v)\Gamma(f,g)|_{\infty,\beta} \leq c_\beta |f|_{\infty,\beta} |g|_{\infty,\beta}$, $\beta \geq 1$.

For the proofs of these assertions, we note that (i) is already known. For part (ii): the "loss terms" are (with $q = q(\omega, |u-v|)$)

$$\Gamma_\ell \equiv \mu^{-1/2}(v) \iint q(f(u)\mu^{1/2}(u)\mu^{1/2}(v)g(v) + f(v)\mu^{1/2}(v)g(u)\mu^{1/2}(u))\,d\omega\,du. \tag{3.138}$$

A typical term is

$$
\begin{aligned}
(3.139)\quad \Gamma_{\ell,t} &\equiv \iint |\omega\cdot(v-u)|\mu^{1/2}(u)f(u)g(v)\,du\,d\omega \\
&= \langle v\rangle^{\beta}g(v)\cdot\langle v\rangle^{-\beta}\iint |\omega\cdot(v-u)|\mu^{1/2}(u)\frac{\langle u\rangle^{\beta}f(u)\,d\omega\,du}{\langle u\rangle^{\beta}} \\
&\le c|g|_{\infty,\beta}|f|_{\infty,\beta}\langle v\rangle^{-\beta}\cdot\int(|v|+|u|)\mu^{1/2}(u)\,du \\
&\le c|g|_{\infty,\beta}|f|_{\infty,\beta}\langle v\rangle^{1-\beta},\ \beta\ge 1.
\end{aligned}
$$

By (3.56), $\nu(v)\ge c|v|$ for large $|v|$. Thus

$$
(3.140)\qquad \nu^{-1}(v)\langle v\rangle^{\beta}\Gamma_{\ell,t}\le c|g|_{\infty,\beta}|f|_{\infty,\beta}\quad\text{for all}\quad v.
$$

The gain terms Γ_g are dominated by

$$
\begin{aligned}
&\left|\mu^{-1/2}(v)\iint q\left[\mu^{1/2}(v')f(v')\mu^{1/2}(u')g(u')+\mu^{1/2}(u')f(u')\mu^{1/2}(v')g(v')\right]du\,d\omega\right| \\
&\le c\iint |\omega\cdot(v-u)|\mu^{1/2}(u)\left(|f(v')g(u')|+|f(u')g(v')|\right)du\,d\omega \\
&\le c\iint |\omega\cdot(v-u)|\mu^{1/2}(u)\left[|f|_{\infty,\beta}|g|_{\infty,\beta}\langle v'\rangle^{-\beta}\langle u'\rangle^{-\beta}\right]du\,d\omega.
\end{aligned}
$$

Now

$$
\begin{aligned}
(3.141)\qquad \langle v'\rangle^{2}\langle u'\rangle^{2} &= (1+|v'|^{2})(1+|u'|^{2}) \\
&\ge 1+|v'|^{2}+|u'|^{2}\ge 1+|v|^{2}=\langle v\rangle^{2}.
\end{aligned}
$$

Thus

$$
\begin{aligned}
(3.142)\qquad |\Gamma_g| &\le c|f|_{\infty,\beta}|g|_{\infty,\beta}\int(|v|+|u|)\mu^{1/2}(u)\langle v\rangle^{-\beta}\,du \\
&\le c|f|_{\infty,\beta}|g|_{\infty,\beta}\langle v\rangle^{1-\beta}\ \text{for}\ \beta\ge 1.
\end{aligned}
$$

As above, we now use $\nu(v)\ge c|v|$ (which is valid for $|v|\ge 1$) and we are done.

Remark. These estimates indicate that the spaces L^{∞}_{β} are the "right ones".

Now we state and prove the key estimate on the nonlinear term $\Gamma(f,g)$.

THEOREM 3.8.1 *Assume the hard-sphere case. Let* $\ell>\frac{3}{2}$, $\beta\ge 1$ *and* $f,g\in\dot{L}^{\infty}_{\beta}(v;H^{\ell}(x))$. *Then*

$$
\nu^{-1}\Gamma(f,g)\in\dot{L}^{\infty}_{\beta}(v,H^{\ell}(x))
$$

and

$$
(3.143)\qquad |\!|\!|\nu^{-1}\Gamma(f,g)|\!|\!|_{\ell,\beta}\le c_{\beta}|\!|\!|f|\!|\!|_{\ell,\beta}|\!|\!|g|\!|\!|_{\ell,\beta}.
$$

Proof. For the hard-sphere case, we have $q(\omega, |u-v|) = \omega \cdot (v-u)$ on S^2_+. We will take $\ell = 2$ for simplicity. Then

$$\begin{aligned}
&\sum_{|\alpha|\le 2} |D^\alpha_x \Gamma_{\text{loss}}(f,g)| \\
&\le c\mu^{-1/2} \iint q\mu^{1/2}(u)\mu^{1/2}(v)[|f(u)g(v)| + |f_x(u)g(v)| \\
&\quad + |f(u)g_x(v)| + |f_{xx}(u)g(v)| + |f_x(u)g_x(v)| + |f(u)g_{xx}(v)|]\, d\omega\, du + \cdots \\
&\equiv L_1 + L_2 + \cdots + L_6 + \text{(6 similar terms with } f \leftrightarrow g).
\end{aligned}$$

Similarly,

$$\sum_{|\alpha|\le 2} |D^\alpha_x \Gamma_{\text{gain}}(f,g)| \le G_1 + G_2 + \cdots + G_6 + \text{(6 similar terms with} f \leftrightarrow g)$$

where e.g.,

$$\begin{aligned}
G_1 &= \mu^{-1/2} \iint q\mu^{1/2}(u')\mu^{1/2}(v')|f(u')||g(v')|\, d\omega\, du \\
G_2 &= \mu^{-1/2} \iint q\mu^{1/2}(u')\mu^{1/2}(v')|f_x(u')||g(v')|\, d\omega\, du,
\end{aligned} \tag{3.144}$$

etc.

Consider the loss terms. We abbreviate $f(u) = f(x,u)$, etc. and will often use the Sobolev inequality $\sup_x |f(x,u)| \le c\|f(\cdot,u)\|_{H^2(x)}$. A typical term L_6 satisfies

$$\begin{aligned}
\|L_6\|^2_{L^2(x)} &\le \int \left| g_{xx}(v) \int \mu^{1/2}(u) q |f(u)|\, d\omega\, du \right|^2 dx \\
&= \int \left| \langle v\rangle^{-\beta}\langle v\rangle^{\beta} |g_{xx}(v)| \int \mu^{1/2}(u) q \langle u\rangle^{-\beta}\langle u\rangle^{\beta} |f(u)|\, d\omega\, du \right|^2 dx \\
&\le c\langle v\rangle^{-2\beta} \cdot \left[\sup_u \langle u\rangle^{\beta} \|f(\cdot,u)\|_{H^2(x)}\right]^2 \cdot \left[\int \mu^{1/2}(u) q \langle u\rangle^{-\beta}\, d\omega\, du\right]^2 \\
&\quad \cdot \langle v\rangle^{2\beta} \|g(\cdot,v)\|^2_{H^2(x)}.
\end{aligned}$$

Now recall that for the hard sphere,

$$\int_{S^2_+} q\, d\omega = \text{(pos. const.)}|v-u|. \tag{3.145}$$

Hence

$$\int \mu^{1/2}(u) q \langle u\rangle^{-\beta}\, d\omega du = c\int |v-u|\mu^{1/2}(u)\langle u\rangle^{-\beta} du \le c(1+|v|). \tag{3.146}$$

Therefore

$$\|L_6\|^2_{L^2(x)} \le c\langle v\rangle^{-2\beta}(1+|v|)^2 |||f|||^2_{2,\beta} \cdot \langle v\rangle^{2\beta} \|g(\cdot,v)\|^2_{H^2(x)}. \tag{3.147}$$

From (3.54), $c_0(1+|v|) \le \nu(v) \le c_1(1+|v|)$. Thus we get

$$\frac{\langle v\rangle^\beta}{\nu(v)}\|L_6\|_{L^2(x)} \le c|\!|\!|f|\!|\!|_{2,\beta} \cdot \langle v\rangle^\beta \|g(\cdot,v)\|_{H^2(x)}. \tag{3.148}$$

Thus with $H^0(x) = L^2(x)$ we have $\nu^{-1}L_6 \in \dot{L}^\infty_\beta(v, H^0(x))$ and

$$|\!|\!|\nu^{-1}L_6|\!|\!|_{0,\beta} \le c|\!|\!|f|\!|\!|_{2,\beta} \cdot |\!|\!|g|\!|\!|_{2,\beta} \tag{3.149}$$

as desired.

Next consider the term L_5 which involves $f_x(u)g_x(v)$. We have

$$L_5 = g_x(v)\int\!\!\int \mu^{1/2}(u) q f_x(u)\, d\omega\, du = c g_x(v)\int |v-u|\mu^{1/2}(u) f_x(u)\, du. \tag{3.150}$$

Therefore

$$\begin{aligned}\|L_5\|^2_{L^2(x)} &= c\int |g_x(v)|^2 \left|\int |v-u|\mu^{1/2}(u) f_x(u)\, du\right|^2 dx \\ &\le c\int |g_x(v)|^2 \left(\int |v-u|^2 \mu^{1/2}(u)\, du\right)\left(\int \mu^{1/2}(u) f_x^2(u)\, du\right) dx.\end{aligned}$$

The middle term here is $0((1+|v|)^2)$. On the last factor we use

$$\begin{aligned}\int \mu^{1/2}(u) f_x^2(u)\, du &\le \left(\int \mu^{1/2}\, du\right)^{1/2}\left(\int \mu^{1/2} f_x^4(u)\, du\right)^{1/2} \\ &\le c\left(\int \mu^{1/2}(u) f_x^4(u)\, du\right)^{1/2}.\end{aligned} \tag{3.151}$$

Hence $\|L_5\|^2_{L^2(x)}$ is dominated by

$$\begin{aligned}& c(1+|v|)^2 \int |g_x(v)|^2 \left(\int \mu^{1/2}(u) f_x^4(u)\, du\right)^{1/2} dx \\ &\le c(1+|v|)^2 \left(\int |g_x(v)|^4\, dx\right)^{1/2}\left(\int\!\!\int \mu^{1/2}(u) f_x^4(u)\, du\, dx\right)^{1/2} \\ &= c(1+|v|)^2 \|g_x(\cdot,v)\|^2_{L^4(x)} \left(\int \mu^{1/2}(u)\|f_x(\cdot,u)\|^4_{L^4(x)}\, du\right)^{1/2} \\ &\le c(1+|v|)^2 \|g(\cdot,v)\|^2_{H^2(x)} \left(\int \mu^{1/2}(u)\|f(\cdot,u)\|^4_{H^2(x)}\, du\right)^{1/2} \\ &= c(1+|v|)^2\langle v\rangle^{-2\beta}\left[\langle v\rangle^\beta \|g(\cdot,v)\|_{H^2(x)}\right]^2 \left[\int \mu^{1/2}(u)\langle u\rangle^{-4\beta}\langle u\rangle^{4\beta}\|f(\cdot,u)\|^4_{H^2(x)}\, du\right]^{\frac{1}{2}}.\end{aligned}$$

It follows that

$$\begin{aligned}&\nu^{-1}(v)\langle v\rangle^\beta \|L_5\|_{L^2(x)} \\ &\le c\langle v\rangle^\beta \|g(\cdot,v)\|_{H^2(x)} \cdot \sup_u \langle u\rangle^\beta \|f(\cdot,u)\|_{H^2(x)} \left[\int \mu^{1/2}(u)\langle u\rangle^{-4\beta} du\right]^{1/4} \\ &\le c|\!|\!|g|\!|\!|_{2,\beta}|\!|\!|f|\!|\!|_{2,\beta}\end{aligned}$$

as desired. All other loss terms can be handled in the same manner.

Now we consider a typical gain term, say G_4. Using $\langle v'\rangle^2\langle u'\rangle^2 \geq \langle v\rangle^2$ from (3.141), we get

$$
\begin{aligned}
\|G_4\|^2_{L^2(x)} &= \int \left| \int q\mu^{1/2}(u) f_{xx}(u') g(v')\, d\omega\, du \right|^2 dx \\
&\leq \langle v\rangle^{-2\beta} \int \left| \int q\mu^{1/2}(u)\langle u'\rangle^{\beta} |f_{xx}(u')| \langle v'\rangle^{\beta} |g(v')|\, d\omega\, du \right|^2 dx \\
&\leq \langle v\rangle^{-2\beta} \int \left(\int q\mu^{1/2}(u)\langle u'\rangle^{2\beta} f^2_{xx}(u')\, d\omega\, du \right) \cdot \\
&\qquad \left(\int q\mu^{1/2}(u)\langle v'\rangle^{2\beta} g^2(v')\, d\omega\, du \right) dx.
\end{aligned}
$$

By the Sobolev inequality we can write

(3.152)
$$
\begin{aligned}
&\|G_4\|^2_{L^2(x)} \\
&\leq c\langle v\rangle^{-2\beta} \left(\int q\mu^{1/2}\langle v'\rangle^{2\beta} \|g(\cdot,v')\|^2_{H^2(x)}\, d\omega\, du \right) \int q\mu^{1/2}\langle u'\rangle^{2\beta} \|f(\cdot,u')\|^2_{H^2(x)}\, d\omega\, du \\
&\leq c\langle v\rangle^{-2\beta} \left[\sup_{v'} \langle v'\rangle^{\beta} \|g(\cdot,v')\|_{H^2(x)} \right]^2 \cdot \left[\sup_{u'} \langle u'\rangle^{\beta} \|f(\cdot,u')\|_{H^2(x)} \right]^2 \cdot \left(\int q\mu^{1/2}\, d\omega\, du \right)^2.
\end{aligned}
$$

Using (3.145) above, we get

$$\int q\mu^{1/2}(u)\, d\omega\, du = c \int |v-u|\mu^{1/2}(u)\, du \leq c(1+|v|). \tag{3.153}$$

Therefore

$$\frac{\langle v\rangle^{\beta}\|G_4\|_{L^2(x)}}{\nu(v)} \leq c|\!|\!| g |\!|\!|_{2,\beta} |\!|\!| f |\!|\!|_{2,\beta} \tag{3.154}$$

as desired. In order to see that e.g.,

$$\nu^{-1}G_4 \in \dot{L}^{\infty}_{\beta}(v, H^0(x)) \tag{3.155}$$

we have this simple lemma:

LEMMA 3.8.1 : *If $|v|$ is large, say $|v|^2 \geq 4$, then either $|u'|^2 \geq |v|^2/4$ or else $|v'|^2 \geq |v|^2/4$.*

Proof. Consider the invariant energy $e = |u|^2 + |v|^2$. If the lemma were false, then

$$|v|^2 \leq e = |u'|^2 + |v'|^2 < \frac{|v|^2}{4} + \frac{|v|^2}{4} = \frac{|v|^2}{2} \tag{3.156}$$

which is a contradiction.

Now we can decompose G_4 as

$$(3.157)\qquad |G_4| \le \int_{\{(\omega,u):|u'|\ge|v|/2\}} q\mu^{1/2}(u)|f_{xx}(u')g(v')|\,d\omega\,du + \int_{\{(\omega,u):|v'|\ge|v|/2\}} q\mu^{1/2}(u)|f_{xx}(u')g(v')|\,d\omega\,du.$$

Each of these terms satisfies an estimate of the form (3.152). Since both $f, g \in \dot{L}^\infty_\beta(v, H^2(x))$, each term tends to 0 as $|v| \to \infty$. The rest of the gain terms can be similarly treated.

3.9. Equations for 13 Moments

Under the Fourier Transform, $v \cdot \nabla_x f \mapsto i(v\cdot\xi)\hat{f}$. Now we regard $v \cdot \xi$ as a linear operator from $N(L) \to$ some subspace $\tilde{W}$ of $L^2(v)$. Consider

$$(3.158)\qquad \phi_1 = 1;\ \phi_{j+1} = v_j\ (j=1,2,3);\ \phi_{j+4} = v_j^2\ (j=1,2,3); \quad \phi_8 = v_1v_2;\ \phi_9 = v_2v_3;\ \phi_{10} = v_3v_1;\ \phi_{j+10} = |v|^2 v_j\ (j=1,2,3).$$

Recall the form of the summational invariants ψ_k:

$$(3.159)\qquad \psi_1 = 1;\ \psi_{j+1} = v_j\ (j=1,2,3);\ \psi_5 = |v|^2.$$

Clearly each ψ_k can be written as a linear combination of $\phi_1, \dots, \phi_7$. Put

$$\tilde{W} = \text{linear span of}\quad \{\mu^{1/2}\phi_k\}_{k=1}^{13}.$$

Note that $v \cdot \xi : N(L) \to \tilde{W}$ for all $\xi \in \mathbb{R}^3$, and that $N(L) \subset \tilde{W}$.

Now we construct an orthonormal basis for $\tilde{W}$. We already have five orthonormal vectors $e_1, \dots, e_5$ from (3.122), (3.124) and (3.131), so we need 8 more. Consider for some $\alpha > 0$

$$(3.160)\qquad \tilde{e}_{j+4} = \alpha\left(v_j^2 - \frac{1}{2}\right)\mu^{1/2} \qquad (j=1,2,3).$$

We claim that these are mutually orthogonal. Indeed, using (3.120) we get e.g.,

$$\begin{aligned}
\alpha^{-2}(\tilde{e}_5, \tilde{e}_6) &= \int \mu\left(v_1^2 - \frac{1}{2}\right)\left(v_2^2 - \frac{1}{2}\right)dv \\
&= \int v_1^2 v_2^2 \mu\,dv - \int v_1^2\mu\,dv + \frac{1}{4}\int \mu\,dv \\
&= \int_0^\infty r^6 e^{-r^2}\,dr \int_{|\omega|=1} \omega_1^2\omega_2^2\,d\omega - \int_0^\infty r^4 e^{-r^2}\,dr \int \omega_1^2\,d\omega + \frac{\pi^{3/2}}{4} \\
&= \frac{3\cdot 5}{16}\sqrt{\pi}\cdot\frac{4\pi}{15} - \frac{3\sqrt{\pi}}{8}\cdot 2\pi\int_0^\pi \cos^2\phi\sin\phi\,d\phi + \frac{1}{4}\pi^{3/2} \\
&= \frac{\pi^{3/2}}{4} - \frac{3}{4}\pi^{3/2}\left[-\frac{1}{3}\cos^3\phi\Big|_0^\pi\right] + \frac{\pi^{3/2}}{4} \\
&= 0.
\end{aligned}$$

To find the normalization constant α, we compute

$$
\begin{aligned}
\|\tilde{e}_{j+4}\|^2 &= \alpha^2 \int \mu \left(v_1^4 - v_1^2 + \frac{1}{4} \right) dv \\
&= \alpha^2 \left[\int_0^\infty r^6 e^{-r^2} dr \int_{|\omega|=1} \omega_1^4 \, d\omega - \int_0^\infty r^4 e^{-r^2} dr \int_{|\omega|=1} \omega_1^2 \, d\omega + \frac{1}{4} \int \mu \, dv \right] \\
&= \alpha^2 \left[\frac{3 \cdot 5}{16} \sqrt{\pi} \cdot \frac{4\pi}{5} - \frac{3}{8} \sqrt{\pi} \cdot 2\pi \int_0^\pi \cos^2 \phi \sin \phi \, d\phi + \frac{1}{4} \pi^{3/2} \right] \\
&= \alpha^2 \left[\frac{3}{4} \pi^{3/2} - \frac{3}{4} \pi^{3/2} \cdot \frac{2}{3} + \frac{1}{4} \cdot \pi^{3/2} \right] \\
&= \frac{\alpha^2}{4} \pi^{3/2} (3 - 2 + 1) = \frac{\alpha^2 \pi^{3/2}}{2}.
\end{aligned}
$$

Thus we take $\alpha = \sqrt{2}\,\pi^{-3/4}$, i.e.,

$$
\tilde{e}_{j+4} = \sqrt{2}\,\pi^{-3/4} \left(v_j^2 - \frac{1}{2} \right) \mu^{1/2} \quad (j = 1, 2, 3). \tag{3.161}
$$

Then the $\{\tilde{e}_{j+4}\}_{j=1}^3$ are orthonormal and

$$
\tilde{e}_5 + \tilde{e}_6 + \tilde{e}_7 = \sqrt{2}\,\pi^{-3/4} \mu^{1/2} \left(|v|^2 - \frac{3}{2} \right) = \sqrt{3}\, e_5, \tag{3.162}
$$

i.e.,

$$
e_5 = \frac{1}{\sqrt{3}} (\tilde{e}_5 + \tilde{e}_6 + \tilde{e}_7). \tag{3.163}
$$

Set $c_1 = \frac{1}{\sqrt{3}}[1, 1, 1]$; choose 2 other orthonormal vectors c_2, c_3 such that $c_1 \times c_2 = c_3$ and $\{c_1, c_2, c_3\}$ forms an orthonormal basis for $\mathbb{R}^3$. Thus

$$
c_3 = \frac{1}{\sqrt{3}} \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 1 & 1 & 1 \\ c_{21} & c_{22} & c_{23} \end{vmatrix} = \frac{1}{\sqrt{3}} \left[c_{23} - c_{22}, c_{21} - c_{23}, c_{22} - c_{21} \right]. \tag{3.164}
$$

Note that $\sum_{j=1}^3 c_{3j} = 0$. It follows that $\sum_{j=1}^3 c_{2j} = 0$ also. To continue the basis computation for $\tilde{W}$, let

$$
\begin{aligned}
e_{k+4} &= \sum_{j=1}^3 c_{kj} \tilde{e}_{j+4} \qquad (k = 2, 3), \\
e_8 &= \alpha_8 v_1 v_2 \mu^{1/2},\ e_9 = \alpha_9 v_2 v_3 \mu^{1/2},\ e_{10} = \alpha_{10} v_3 v_1 \mu^{1/2}, \\
e_{j+10} &= \alpha_{11} (|v|^2 - \beta_{10}) v_j \mu^{1/2} \qquad (j = 1, 2, 3).
\end{aligned} \tag{3.165}
$$

Next we perform a brief check of orthogonality. We begin with

$$
\begin{aligned}
(e_6, e_5) &= \left(\frac{1}{\sqrt{3}} (\tilde{e}_5 + \tilde{e}_6 + \tilde{e}_7), c_{21} \tilde{e}_5 + c_{22} \tilde{e}_6 + c_{23} \tilde{e}_7 \right) \\
&= \frac{1}{\sqrt{3}} (c_{21} + c_{22} + c_{23}) = 0.
\end{aligned} \tag{3.166}
$$

Similarly $(e_7, e_5) = 0$. Clearly $(e_6, e_j) = 0$, $j = 2, 3, 4$ because the integrand in each case is odd. Next,

$$(e_6, e_1) = (c_{21}\tilde{e}_5 + c_{22}\tilde{e}_6 + c_{23}\tilde{e}_7, e_1) \tag{3.167}$$

Since $(\tilde{e}_{j+4}, e_1)$ are all equal, $j = 1, 2, 3$, $(e_6, e_1) = 0$ because $c_{21} + c_{22} + c_{23} = 0$. Similarly $(e_7, e_1) = 0$.

Clearly, e_8, e_9, e_{10} are orthogonal to e_ℓ, $1 \le \ell \le 7$, and for $\ell = 11, 12, 13$. Furthermore e_{j+10} $(j = 1, 2, 3)$ are clearly orthogonal to e_ℓ except for $\ell = 2, 3, 4$. For these cases, we will choose β_{10}. We force $(e_{11}, e_2) = 0$ via

$$\int v_1^2(|v|^2 - \beta_{10})\mu\, dv = 0 : \tag{3.168}$$

$$\int_0^\infty r^6 e^{-r^2}\, dr \int_{|\omega|=1} \omega_1^2\, d\omega - \beta_{10} \int_0^\infty r^4 e^{-r^2}\, dr \int_{|\omega|=1} \omega_1^2\, d\omega = 0,$$

or

$$\frac{4\pi}{3}\left[\frac{3 \cdot 5}{16}\sqrt{\pi} - \beta_{10} \cdot \frac{3}{8}\sqrt{\pi}\right] = 0. \tag{3.169}$$

Therefore $\beta_{10} = \frac{5}{2}$, i.e.,

$$e_{j+10} = \alpha_{11}\left(|v|^2 - \frac{5}{2}\right) v_j \mu^{1/2} \qquad (j = 1, 2, 3). \tag{3.170}$$

Then clearly also

$$(e_{12}, e_j) = (e_{13}, e_j) = 0, \qquad j = 3, 4. \tag{3.171}$$

Lastly we consider the normalization constants. $e_1, \ldots, e_7$ already have norm 1. We will have $\alpha_8 = \alpha_9 = \alpha_{10}$ with

$$\begin{aligned} 1 = \|e_8\|^2 &= \alpha_8^2 \int v_1^2 v_3^2 \mu\, dv = \alpha_8^2 \int_0^\infty r^6 e^{-r^2} dr \int_{|\omega|=1} \omega_1^2 \omega_3^2\, d\omega \\ &= \alpha_8^3 \left(\frac{3 \cdot 5}{16}\sqrt{\pi}\right) \cdot \frac{4\pi}{15} = \frac{\alpha_8^2 \pi^{3/2}}{4}. \end{aligned} \tag{3.172}$$

Therefore we take

$$\alpha_8 = \alpha_9 = \alpha_{10} = 2\pi^{-3/4}. \tag{3.173}$$

For α_{11}, we require

$$\begin{aligned} 1 = \|e_{11}\|^2 &= \alpha_{11}^2 \int \left(|v|^2 - \frac{5}{2}\right)^2 v_1^2 \mu\, dv \\ &= \alpha_{11}^2 \int_0^\infty r^4 \left(r^2 - \frac{5}{2}\right)^2 e^{-r^2}\, dr \int_{|\omega|=1} \omega_1^2\, d\omega \\ &= \alpha_{11}^2 \cdot \frac{4\pi}{3} \int_0^\infty \left(r^8 - 5r^6 + \frac{25}{4} r^4\right) e^{-r^2}\, dr \\ &= \frac{\alpha_{11}^2 \cdot 4\pi}{3} \cdot \sqrt{\pi}\left[\frac{3 \cdot 5 \cdot 7}{32} - \frac{5 \cdot 3 \cdot 5}{16} + \frac{25}{4} \cdot \frac{3}{8}\right] \\ &= \alpha_{11}^2 \pi^{3/2} \cdot \frac{5}{4}. \end{aligned} \tag{3.174}$$

Therefore

$$\alpha_{11} = \frac{2}{\sqrt{5}}\pi^{-3/4}. \tag{3.175}$$

Therefore we can conclude that an orthonormal basis for $\tilde{W}$ is given by $\{e_j\}_{j=1}^{13}$, where

$$\tilde{e}_{j+4} = \sqrt{2}\,\pi^{-3/4}\left(v_j^2 - \frac{1}{2}\right)\mu^{1/2} \quad (j = 1,\, 2,\, 3)$$

and

$$\begin{aligned}
&e_1 = \pi^{-3/4}\mu^{1/2};\; e_{j+1} = \sqrt{2}\,\pi^{-3/4}v_j\mu^{1/2} \qquad (j = 1.2.3): \\
&e_5 = \sqrt{\frac{2}{3}}\,\pi^{-3/4}\left(|v|^2 - \frac{3}{2}\right)\mu^{1/2}; \\
&e_6 = \sum_{j=1}^{3} c_{2j}\tilde{e}_{j+4};\; e_7 = \sum_{j=1}^{3} c_{3j}\tilde{e}_{j+4}; \\
&e_8 = 2\pi^{-3/4}v_1v_2\mu^{1/2};\; e_9 = 2\pi^{-3/4}v_3v_2\mu^{1/2};\; e_{10} = 2\pi^{-3/4}v_1v_3\mu^{1/2}; \\
&e_{j+10} = \frac{2}{\sqrt{5}}\,\pi^{-3/4}\left(|v|^2 - \frac{5}{2}\right)v_j\mu^{1/2} \qquad (j = 1, 2, 3).
\end{aligned} \tag{3.176}$$

Let P be the orthogonal projection from $L^2(v)$ into $\tilde{W}$:

$$Pf = \sum_{k=1}^{13}(f, e_k)e_k. \tag{3.177}$$

In what follows we compute the equation satisfied by Pf. From

$$f_t + v\cdot\nabla_x f + L[f] = \Gamma(f, f) \tag{3.178}$$

we compute, for $W_k = (f, e_k) \quad (k = 1, \ldots, 13)$,

$$\frac{\partial W_k}{\partial t} = (f_t, e_k) = -(v\cdot\nabla_x f. e_k) - (L[f]. e_k) + (\Gamma(f. f). e_k). \tag{3.179}$$

Split f using

$$f = Pf + (I - P)f. \tag{3.180}$$

Then

(3.181)

$$\begin{aligned}
(v\cdot\nabla_x f, e_k) &= \sum_\ell\left(v_\ell\frac{\partial f}{\partial x_\ell}, e_k\right) = \sum_\ell\left(\frac{\partial f}{\partial x_\ell}, v_\ell e_k\right) \\
&= \sum_\ell \frac{\partial}{\partial x_\ell}(f, v_\ell e_k) = \sum_\ell \frac{\partial}{\partial x_\ell}(Pf + (I-P)f, v_\ell e_k) \\
&= \sum_\ell \frac{\partial}{\partial x_\ell}\left(\sum_j (f, e_j)e_j, v_\ell e_k\right) + \sum_\ell \frac{\partial}{\partial x_\ell}((I-P)f. v_\ell e_k) \\
&= \sum_\ell \frac{\partial}{\partial x_\ell}\left(\sum_j W_j e_j, v_\ell e_k\right) + \sum_\ell \frac{\partial}{\partial x_\ell}((I-P)f, v_\ell e_k).
\end{aligned}$$

For $\ell = 1, 2, 3$ define a matrix V^ℓ by

$$(V^\ell)_{kj} = (v_\ell e_k, e_j) \qquad (k, j = 1, \dots, 13). \tag{3.182}$$

Then the first term in (3.181) is

$$\begin{aligned}\sum_\ell \frac{\partial}{\partial x_\ell}\left(\sum_j W_j e_j, v_\ell e_k\right) &= \sum_\ell \frac{\partial}{\partial x_\ell}\sum_j (e_j, v_\ell e_k) W_j \\ &= \sum_\ell \frac{\partial}{\partial x_\ell}\left(\sum_j V^\ell_{kj} W_j\right) \\ &= \sum_\ell \frac{\partial}{\partial x_\ell}(V^\ell W)_k \\ &= \sum_\ell \left(V^\ell \frac{\partial W}{\partial x_\ell}\right)_k .\end{aligned} \tag{3.183}$$

Now we compute $(L[f], e_k)$ similarly:

$$\begin{aligned}(L[f], e_k) &= (L[Pf + (I-P)f], e_k) \\ &= \left(L\left[\sum_j W_j e_j\right], e_k\right) + (L[(I-P)f], e_k) \\ &= \left(\sum_j W_j L[e_j], e_k\right) + (L[(I-P)f], e_k).\end{aligned} \tag{3.184}$$

For $k, j = 1, \dots, 13$ define

$$\overline{L}_{kj} = (L[e_j], e_k).$$

Then the first term in (3.184) is

$$\sum_j W_j (L[e_j], e_k) = \sum_j \overline{L}_{kj} W_j = (\overline{L} W)_k. \tag{3.185}$$

For the collision term we compute

$$\begin{aligned}&(\Gamma(f, f), e_k) \\ &= (\Gamma(Pf + (I-P)f, Pf + (I-P)f), e_k) \\ &= (\Gamma(Pf, Pf), e_k) + 2(\Gamma((I-P)f, Pf), e_k) + (\Gamma((I-P)f, (I-P)f), e_k).\end{aligned}$$

For the first term we have

$$\begin{aligned}\Gamma(Pf, Pf) &= \Gamma\left(\sum_\ell W_\ell e_\ell, \sum_m W_m e_m\right) \\ &= \mu^{-1/2}(v) Q^*(\mu^{1/2} Pf, \mu^{1/2} Pf) \\ &= \mu^{-1/2}(v) \int\!\!\int q(\omega, |u-v|)[\mu^{1/2}(v') Pf(v') \mu^{1/2}(u') Pf(u') - \\ &\qquad \mu^{1/2}(v) Pf(v) \mu^{1/2}(u) Pf(u)]\, d\omega\, du.\end{aligned}$$

We call the "loss" term $\Gamma_-(Pf, Pf)$ and calculate

$$\begin{aligned}
&(\Gamma_-(Pf, Pf), e_k) \\
&= \iint \mu^{1/2}(u) Pf(u) \left(\mu^{-1/2} q \cdot \mu^{1/2} Pf, e_k\right) d\omega\, du \\
&= \iint \mu^{1/2}(u) \sum_\ell W_\ell(x) e_\ell(u) \left(\mu^{-1/2} q \mu^{1/2} \sum_m W_m(x) e_m(v), e_k\right) d\omega\, du \\
&= \sum_{m,\ell} W_m W_\ell \iint \mu^{1/2}(u) e_\ell(u) \left(\mu^{-1/2} q \cdot \mu^{1/2} e_m, e_k\right) \\
&= \sum_{m,\ell} W_m W_\ell (\Gamma_-(e_\ell, e_m), e_k).
\end{aligned}$$

A similar calculation works for the gain term $\Gamma_+(Pf, Pf)$.

Now write

$$(\Gamma(Pf, Pf), e_k) = q_k(W) \equiv \sum_{\ell,m=1}^{13} q^k_{\ell m} W_\ell W_m \tag{3.186}$$

where

$$q^k_{\ell m} = \{(\Gamma\,(e_\ell, e_m), e_k)\}_{1 \le k,\ell,m \le 13}\,. \tag{3.187}$$

In conclusion, we have the following: let

$$W = \begin{bmatrix} W_1 \\ \vdots \\ W_{13} \end{bmatrix}, \quad W_k = (f, e_k). \tag{3.188}$$

Then

$$\frac{\partial W}{\partial t} + \sum_j V^j \frac{\partial W}{\partial x_j} + \bar{L} W = q(W) + r(f, \nabla f) \tag{3.189}$$

where $r(\cdot,\cdot)$ denotes all the remainder terms.

In particular, when $f = Pf$, we have

$$\frac{\partial W}{\partial t} + \sum_{j=1}^{3} V^j \frac{\partial W}{\partial x_j} + \bar{L} W = q(W) \tag{3.190}$$

which is a symmetric hyperbolic system with constant coefficients.

3.10. Computation of the Coefficient Matrices

For $\xi \in \mathbb{R}^3$, let

$$V(\xi) = \sum_{j=1}^{3} V^j \xi_j = \left\{((v \cdot \xi) e_k, e_\ell)\right\}_{k,\ell=1}^{13}. \tag{3.191}$$

Thus V is **symmetric**. The 13×5 matrix given by the first 5 columns gives the representation of $v \cdot \xi$ as a linear operator from $N(L) \to \tilde{W}$. Decompose $V(\xi)$ as

$$V(\xi) = \begin{bmatrix} V_{11}(\xi)^{5\times 5} & V_{12}(\xi)^{5\times 8} \\ V_{21}(\xi)^{8\times 5} & V_{22}(\xi)^{8\times 8} \end{bmatrix}. \tag{3.192}$$

So $V_{11}^T = V_{11}$. $V_{12}^T = V_{21}$, $V_{22}^T = V_{22}$.

Now we compute $V_{11}(\xi)$ whose matrix elements are given by

$$\Big\{((v \cdot \xi)e_k, e_\ell)\Big\}_{1\le k,\ell\le 5}. \tag{3.193}$$

Take $k = 1$ and write $V_{11} \equiv \tilde{V}$ for brevity. Then $\tilde{V}_{11} = \tilde{V}_{15} = 0$ because the integrands are odd functions of v, and

$$\begin{aligned} \tilde{V}_{12} &= ((v \cdot \xi)e_1, e_2) = \pi^{-3/4} \cdot \sqrt{2}\,\pi^{-3/4}(\mu^{1/2}(v \cdot \xi), v_1\mu^{1/2}) \\ &= 2^{1/2}\pi^{-3/2}\int \xi_1 \cdot v_1^2 \mu \, dv = 2^{1/2}\pi^{-3/2}\xi_1 \int_0^\infty r^4 e^{-r^2}\,dr \int_{|\omega|=1} \omega_1^2\,d\omega \\ &= 2^{1/2}\pi^{-3/2}\xi_1 \cdot \frac{4\pi}{3} \cdot \frac{3}{8}\sqrt{\pi} = \frac{\xi_1}{\sqrt{2}}. \end{aligned}$$

Similarly

$$\tilde{V}_{13} = \frac{1}{\sqrt{2}}\xi_2, \quad \tilde{V}_{14} = \frac{1}{\sqrt{2}}\xi_3. \tag{3.194}$$

Now taking $k = 2$ we have

$$\tilde{V}_{2\ell} = ((v \cdot \xi)e_2, e_\ell) \qquad (\ell = 1, \dots, 5). \tag{3.195}$$

Recall that

$$e_2 = \sqrt{2}\,\pi^{-3/4}v_1\mu^{1/2}. \tag{3.196}$$

Thus

$$\tilde{V}_{22} = \tilde{V}_{23} = \tilde{V}_{24} = 0. \tag{3.197}$$

and

$$\tilde{V}_{21} = \int (v \cdot \xi)e_2 \cdot e_1\,dv = \pi^{-3/4} \cdot \sqrt{2}\,\pi^{-3/4}\int (v \cdot \xi)\mu v_1\,dv = \frac{\xi_1}{\sqrt{2}} \tag{3.198}$$

by the above.

In a similar manner we then get

$$V_{11}(\xi) = \begin{bmatrix} 0 & \frac{\xi_1}{\sqrt{2}} & \frac{\xi_2}{\sqrt{2}} & \frac{\xi_3}{\sqrt{2}} & 0 \\ \frac{\xi_1}{\sqrt{2}} & 0 & 0 & 0 & \frac{\xi_1}{\sqrt{3}} \\ \frac{\xi_2}{\sqrt{2}} & 0 & 0 & 0 & \frac{\xi_2}{\sqrt{3}} \\ \frac{\xi_3}{\sqrt{2}} & 0 & 0 & 0 & \frac{\xi_3}{\sqrt{3}} \\ 0 & \frac{\xi_1}{\sqrt{3}} & \frac{\xi_2}{\sqrt{3}} & \frac{\xi_3}{\sqrt{3}} & 0 \end{bmatrix}. \tag{3.199}$$

Next we compute $V_{21}(\xi)$ whose elements are given by

$$V_{21}(\xi) = \left\{((v\cdot\xi)e_k, e_\ell)\right\}_{\substack{k=6,\dots,13\\ \ell=1,\dots,5}}. \tag{3.200}$$

Again call $V_{21} = \tilde{V}$ for brevity.

Take $k = 6$: then the first row of $\tilde{V}$ has the elements

$$\int (v\cdot\xi)e_6 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}^T dv. \tag{3.201}$$

Recall that

$$e_6 = c_{21}\tilde{e}_5 + c_{22}\tilde{e}_6 + c_{23}\tilde{e}_7, \tag{3.202}$$

where

$$\tilde{e}_{j+4} = \sqrt{2}\,\pi^{-3/4}\left(v_j^2 - \frac{1}{2}\right)\mu^{1/2} \qquad (j = 1, 2, 3). \tag{3.203}$$

Thus

$$\tilde{V}_{11} = \tilde{V}_{15} = 0 \tag{3.204}$$

and

$$\begin{aligned} \tilde{V}_{12} &= \int (v\cdot\xi)e_6 e_2\, dv \\ & \int (v\cdot\xi)\sum_{j=1}^{3} c_{2j}\tilde{e}_{j+4}\cdot\sqrt{2}\,\pi^{-3/4}v_1\mu^{1/2}\, dv \\ &= \sqrt{2}\cdot\pi^{-3/4}\cdot\sqrt{2}\,\pi^{-3/4}\xi_1\sum_{j=1}^{3} c_{2j}\int v_1^2\left(v_j^2 - \frac{1}{2}\right)\mu\, dv. \end{aligned} \tag{3.205}$$

Now by direct computation,

$$\int v_1^2 v_j^2 \mu\, dv = \frac{1}{2}\int v_1^2\mu\, dv \qquad (j \neq 1). \tag{3.206}$$

Thus

$$\begin{aligned} \tilde{V}_{12} &= 2\pi^{-3/2}\xi_1 c_{21}\int\left(v_1^4 - \frac{v_1^2}{2}\right)\mu\, dv \\ &= 2\pi^{-3/2}c_{21}\xi_1\left[\int_0^\infty r^6 e^{-r^2}dr\int\omega_1^4\, d\omega - \frac{1}{2}\int_0^\infty r^4 e^{-r^2}dr\cdot\int\omega_1^2\, d\omega\right] \\ &= 2\pi^{-3/2}c_{21}\xi_1\left[\frac{3\cdot 5}{16}\sqrt{\pi}\cdot\frac{4\pi}{5} - \frac{1}{2}\cdot\frac{3\sqrt{\pi}}{8}\cdot\frac{4\pi}{3}\right] \\ &= 2c_{21}\xi_1\left[\frac{12}{16} - \frac{1}{4}\right] = c_{21}\xi_1. \end{aligned}$$

Hence the first row is given by

$$\begin{bmatrix}0 & c_{21}\xi_1 & c_{22}\xi_2 & c_{23}\xi_3 & 0\end{bmatrix}. \tag{3.207}$$

Similarly, for $k = 7$ the second row is

$$\begin{bmatrix}0 & c_{31}\xi_1 & c_{32}\xi_2 & c_{33}\xi_3 & 0\end{bmatrix}. \tag{3.208}$$

Next, let $k = 8$: the third row has elements

$$\int (v \cdot \xi) e_8 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}^T dv, \quad \text{where} \quad e_8 = 2\pi^{-3/4} v_1 v_2 \mu^{1/2}. \tag{3.209}$$

Therefore $\tilde{V}_{31} = \tilde{V}_{35} = 0$ and

$$\begin{aligned} \tilde{V}_{32} &= \int (v \cdot \xi) e_8 e_2 \, dv \\ &= 2\pi^{-3/4} \cdot \sqrt{2}\, \pi^{-3/4} \int (v \cdot \xi) v_1 v_2 v_1 \mu \, dv \\ &= 2^{3/2} \pi^{-3/2} \xi_2 \int v_1^2 v_2^2 \mu \, dv \\ &= 2^{3/2} \pi^{-3/2} \xi_2 \int_0^\infty r^6 e^{-r^2} \, dr \cdot \int_{|\omega|=1} \omega_1^2 \omega_2^2 \, d\omega \\ &= 2^{3/2} \pi^{-3/2} \xi_2 \cdot \frac{3 \cdot 5\sqrt{\pi}}{16} \cdot \frac{4\pi}{15} \\ &= \frac{2^{3/2} \xi_2}{4} \\ &= \frac{\xi_2}{\sqrt{2}}. \end{aligned} \tag{3.210}$$

Similarly

$$\tilde{V}_{33} = \int (v \cdot \xi) e_8 e_3 \, dv = \frac{\xi_1}{\sqrt{2}}, \quad \tilde{V}_{34} = \int (v \cdot \xi) e_8 e_4 \, dv = 0. \tag{3.211}$$

Thus the third row is

$$\begin{bmatrix}0 & \dfrac{\xi_2}{\sqrt{2}} & \dfrac{\xi_1}{\sqrt{2}} & 0 & 0\end{bmatrix}. \tag{3.212}$$

When $k = 9$, the fourth row is

$$\int (v \cdot \xi) e_9 \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}^T dv, \quad \text{where} \quad e_9 = 2\pi^{-3/4} v_2 v_3 \mu^{1/2}. \tag{3.213}$$

We get for the fourth row

$$\begin{bmatrix} 0 & 0 & \frac{\xi_3}{\sqrt{2}} & \frac{\xi_2}{\sqrt{2}} & 0 \end{bmatrix} \tag{3.214}$$

and similarly

$$\begin{bmatrix} 0 & \frac{\xi_3}{\sqrt{2}} & 0 & \frac{\xi_1}{\sqrt{2}} & 0 \end{bmatrix} \tag{3.215}$$

for the fifth row.

For row six ($k = 11$), the elements are

$$\int (v \cdot \xi) e_{11} \begin{bmatrix} e_1 \\ e_2 \\ e_3 \\ e_4 \\ e_5 \end{bmatrix}^T dv, \quad \text{where} \quad e_{11} = 2 \cdot 5^{-1/2} \pi^{-3/4} \left(|v|^2 - \frac{5}{2} \right) v_1 \mu^{1/2}. \tag{3.216}$$

Therefore

$$\tilde{V}_{61} = \int (v \cdot \xi) e_{11} e_1 \, dv = c \sum_j \xi_j \int v_j \mu^{1/2} e_{11} \, dv \tag{3.217}$$

for some constant c. But

$$\int v_j \mu^{1/2} e_{11} \, dv = c(e_{j+1}, e_{11}) = 0 \qquad (j = 1, 2, 3) \tag{3.218}$$

and thus $\tilde{V}_{61} = 0$. Similarly $\tilde{V}_{62} = 0$. Clearly $\tilde{V}_{63} = \tilde{V}_{64} = 0$ and

$$\begin{aligned} \tilde{V}_{65} &= \int (v \cdot \xi) e_{11} e_5 \, dv \\ &= 2 \cdot 5^{-1/2} \pi^{-3/4} \cdot \sqrt{\frac{2}{3}} \pi^{-3/4} \int (v \cdot \xi) \left(|v|^2 - \frac{5}{2} \right) v_1 \cdot \left(|v|^2 - \frac{3}{2} \right) \mu \, dv \\ &= 2^{3/2} 3^{-1/2} 5^{-1/2} \pi^{-3/2} \xi_1 \int \mu v_1^2 \left(|v|^2 - \frac{5}{2} \right) \left(|v|^2 - \frac{3}{2} \right) dv. \end{aligned}$$

For the integral

$$I \equiv \int \mu v_1^2 \left(|v|^4 - 4|v|^2 + \frac{15}{4} \right) dv \tag{3.219}$$

we get by direct computation the value $(5/4)\pi^{3/2}$. Hence

$$\begin{aligned} \tilde{V}_{65} &= 2^{3/2} 3^{-1/2} 5^{-1/2} \pi^{-3/2} \xi_1 \cdot \frac{5}{4} \pi^{3/2} \\ &= 2^{-1/2} 3^{-1/2} 5^{1/2} \xi_1 \\ &= \sqrt{\frac{5}{6}} \xi_1. \end{aligned} \tag{3.220}$$

The computation of the remaining rows is similar.

Hence $V_{21}(\xi)$ has the form

$$V_{21}(\xi) = \begin{bmatrix} 0 & c_{21}\xi_1 & c_{22}\xi_2 & c_{23}\xi_3 & 0 \\ 0 & c_{31}\xi_1 & c_{32}\xi_2 & c_{33}\xi_3 & 0 \\ 0 & \frac{\xi_2}{\sqrt{2}} & \frac{\xi_1}{\sqrt{2}} & 0 & 0 \\ 0 & 0 & \frac{\xi_3}{\sqrt{2}} & \frac{\xi_2}{\sqrt{2}} & 0 \\ 0 & \frac{\xi_3}{\sqrt{2}} & 0 & \frac{\xi_1}{\sqrt{2}} & 0 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{6}}\,\xi_1 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{6}}\,\xi_2 \\ 0 & 0 & 0 & 0 & \sqrt{\frac{5}{6}}\,\xi_3 \end{bmatrix}$$

3.11. Compensating Functions

We begin with the definition of Kawashima's Compensating Functions [38].

DEFINITION 3.11.1 Let $R(\omega), \omega \in S^2$ be a real 13×13 matrix. $R(\omega)$ is called a **compensating function** for the system of 13 moments (3.190)

$$W_t + \sum_1^3 V^j \frac{\partial W}{\partial x_j} + \bar{L}W = q(W)$$

if

(i) $R(\cdot) \in C^\infty(S^2)$, $R(-\omega) = -R(\omega)$ for all $\omega \in S^2$,

(ii) $R^*(\omega) = -R(\omega)$,

(iii) There is a $\delta > 0$ such that

$$\Re\langle R(\omega)V(\omega)W, W\rangle + \langle \bar{L}W, W\rangle \geq \delta|W|^2 \tag{3.221}$$

for all $W \in \mathbb{C}^{13}, \omega \in S^2$. (Here $\langle\cdot\rangle$ = inner product on $\mathbb{C}^{13}$).

For the Boltzmann equation (BE) itself, Kawashima's definition is as follows.

DEFINITION 3.11.2 Let $S(\omega), \omega \in S^2$ be a bounded linear operator on $L^2(v)$. S is called a **compensating function** for the (BE) if

(i) $S(\cdot)$ is C^∞ on S^2 with values in the space of bounded linear operators on $L^2(v)$, and $S(-\omega) = -S(\omega)$ for all $\omega \in S^2$,

(ii) $iS(\omega)$ is self-adjoint on $L^2(v)$, for all $\omega \in S^2$,

(iii) There exists $\delta > 0$ such that

(3.222)
$$\Re(S(\omega)(\omega \cdot v)f, f) + (L[f], f) \geq \delta\|f\|^2 \quad \text{for all} \quad f \in L_1^2(v),\ \omega \in S^2.$$

Now we introduce the notation

$$W = [W_1, W_2, \ldots, W_{13}]^T; \tag{3.223}$$

$$W_I = [W_1, \cdots, W_5]^T; \quad W_{II} = [W_6, \cdots, W_{13}]^T.$$

LEMMA 3.11.1 *There exist* 13×13 *real constant entry skew–symmetric matrices* R^j $(j = 1, 2, 3)$ *such that for*

$$R(\omega) \equiv \sum_{j=1}^{3} R^j \omega_j \tag{3.224}$$

we have

$$\Re\langle R(\omega)V(\omega)W, W\rangle \geq c_1|W_I|^2 - C_1|W_{II}|^2 \tag{3.225}$$

for some $c_1 > 0$, *all* $W \in \mathbb{C}^{13}, \omega \in S^2$. *Here* $\langle\ \rangle$ *is the inner product on* $\mathbb{C}^{13}$.

Proof. Let

$$R(\xi) \equiv \sum_{j=1}^{3} R^j \xi_j = \begin{bmatrix} \alpha\overline{R}_{11}(\xi)^{5\times 5} & V_{12}(\xi)^{5\times 8} \\ -V_{21}(\xi)^{8\times 5} & 0^{8\times 8} \end{bmatrix}. \tag{3.226}$$

Here $\alpha > 0$ is a constant to be specified later, and

$$\overline{R}_{11}(\xi) = \begin{bmatrix} 0 & \xi_1 & \xi_2 & \xi_3 & 0 \\ -\xi_1 & 0 & 0 & 0 & 0 \\ -\xi_2 & 0 & 0 & 0 & 0 \\ -\xi_3 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{3.227}$$

Thus each R^j is 13×13 real skew–symmetric with constant entries.

Call

$$\begin{aligned} U(\xi) &= R(\xi)V(\xi) \\ &= \begin{bmatrix} \alpha\overline{R}_{11} & V_{12} \\ -V_{21} & 0 \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix} \\ &= \begin{bmatrix} \alpha\overline{R}_{11}V_{11} + V_{12}V_{21} & \alpha\overline{R}_{11}V_{12} + V_{12}V_{22} \\ -V_{21}V_{11} & -V_{21}V_{12} \end{bmatrix}. \end{aligned} \tag{3.228}$$

Thus

$$R(\omega)V(\omega)W = \begin{bmatrix} (\alpha\overline{R}_{11}V_{11} + V_{12}V_{21})W_I + (\alpha\overline{R}_{11}V_{12} + V_{12}V_{22})W_{II} \\ -V_{21}V_{11}W_I - V_{21}V_{12}W_{II} \end{bmatrix} \tag{3.229}$$

so that

$$\begin{aligned} \langle R(\omega)V(\omega)W, W\rangle &= \langle(\alpha\overline{R}_{11}V_{11} + V_{12}V_{21})W_I, W_I\rangle \\ &\quad + \langle(\alpha\overline{R}_{11}V_{12} + V_{12}V_{22})W_{II}, W_I\rangle \\ &\quad - \langle V_{21}V_{11}W_I, W_{II}\rangle - \langle V_{21}V_{12}W_{II}, W_{II}\rangle. \end{aligned} \tag{3.230}$$

Now suppose that we can show that the real part of the first term is at least $c|W_I|^2$ for some $c > 0$ and appropriately chosen α. The moduli of the second and third terms do not exceed $c|W_I||W_{II}| \le \epsilon|W_I|^2 + c_\epsilon|W_{II}|^2$ for any $\epsilon > 0$. The last term is of the order of $|W_{II}|^2$. The lemma then follows.

Thus we consider

$$U_{11}(\xi) \equiv (\alpha \overline{R}_{11}V_{11} + V_{12}V_{21})(\xi). \tag{3.231}$$

Now

$$\overline{R}_{11}V_{11} = \frac{1}{\sqrt{2}}\begin{bmatrix} |\xi|^2 & 0 & 0 & 0 & \frac{\sqrt{2}}{\sqrt{3}}|\xi|^2 \\ 0 & -\xi_1^2 & -\xi_1\xi_2 & -\xi_1\xi_3 & 0 \\ 0 & -\xi_1\xi_2 & -\xi_2^2 & -\xi_2\xi_3 & 0 \\ 0 & -\xi_1\xi_3 & -\xi_2\xi_3 & -\xi_3^2 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.232}$$

It follows that

$$\begin{aligned}\sqrt{2}\,\Re\langle \overline{R}_{11}(\omega)V_{11}(\omega)W_I, W_I\rangle = \Re\Big\{&\overline{W}_1\Big(W_1 + \frac{\sqrt{2}}{\sqrt{3}}W_5\Big) \\ &+ \overline{W}_2(-\omega_1^2W_2 - \omega_1\omega_2W_3 - \omega_1\omega_3W_4) \\ &+ \overline{W}_3(-\omega_1\omega_2W_2 - \omega_2^2W_3 - \omega_2\omega_3W_4) \\ &+ \overline{W}_4(-\omega_1\omega_3W_2 - \omega_2\omega_3W_3 - \omega_3^2W_4)\Big\} \\ \ge c_2|W_1|^2 - C_2\sum_{k=2}^{5}|W_k|^2&\end{aligned}$$

for some $c_2 > 0$. Next we write

$$\langle V_{12}(\omega)V_{21}(\omega)W_I, W_I\rangle = \Big|V_{21}(\omega)W_I\Big|^2 \tag{3.233}$$

and compute directly

$$V_{21}(\omega)W_I = \begin{bmatrix} c_{21}\omega_1W_2 + c_{22}\omega_2W_3 + c_{23}\omega_3W_4 \\ c_{31}\omega_1W_2 + c_{32}\omega_2W_3 + c_{33}\omega_3W_4 \\ \frac{1}{\sqrt{2}}\omega_2W_2 + \frac{1}{\sqrt{2}}\omega_1W_3 \\ \frac{1}{\sqrt{2}}\omega_3W_3 + \frac{1}{\sqrt{2}}\omega_2W_4 \\ \frac{1}{\sqrt{2}}\omega_3W_2 + \frac{1}{\sqrt{2}}\omega_1W_4 \\ \sqrt{\frac{5}{6}}\ \omega_1W_5 \\ \sqrt{\frac{5}{6}}\ \omega_2W_5 \\ \sqrt{\frac{5}{6}}\ \omega_3W_5 \end{bmatrix}$$

To be specific, we now choose

$$\begin{aligned} c_2 &= \left[\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0\right], \\ c_3 &= \sqrt{\frac{2}{3}}\left[\frac{1}{2}, \frac{1}{2}, -1\right]. \end{aligned} \tag{3.234}$$

For the sum of the squares of the first two rows, we get directly

$$
\begin{aligned}
&|c_{21}\omega_1 W_2 + c_{22}\omega_2 W_3 + c_{23}\omega_3 W_4|^2 + |c_{31}\omega_1 W_2 + c_{32}\omega_2 W_3 + c_{33}\omega_3 W_4|^2 \\
&= \frac{1}{2}\,|\omega_1 W_2 - \omega_2 W_3|^2 + \frac{1}{6}\,|\omega_1 W_2 + \omega_2 W_3 - 2\omega_3 W_4|^2 \\
&= \Re\,\frac{1}{2}[\omega_1^2|W_2|^2 + \omega_2^2|W_3|^2 - 2\omega_1\omega_2 W_2\overline{W}_3] \\
&\quad + \Re\,\frac{1}{6}[\omega_1^2|W_2|^2 + \omega_2^2|W_3|^2 + 4\omega_3^2|W_4|^2 + 2\omega_1\omega_2 W_2\overline{W}_3 - 4\omega_1\omega_3 W_2\overline{W}_4 - 4\omega_2\omega_3 W_3\overline{W}_4] \\
&= \frac{2}{3}(\omega_1^2|W_2|^2 + \omega_2^2|W_3|^2 + \omega_3^2|W_4|^2) - \frac{2}{3}\Re\,(\omega_1\omega_2 W_2\overline{W}_3 + \omega_2\omega_3 W_3\overline{W}_4 + \omega_3\omega_1 W_4\overline{W}_2).
\end{aligned}
$$

Hence $|V_{21}(\omega)W_I|^2$ is equal to

$$
\begin{aligned}
&\frac{2}{3}\Re\,[\omega_1^2|W_2|^2 + \omega_2^2|W_3|^2 + \omega_3^2|W_4|^2 - \omega_1\omega_2 W_2\overline{W}_3 - \omega_2\omega_3 W_3\overline{W}_4 - \omega_3\omega_1 W_4\overline{W}_2] \\
&\quad + \Re\,\frac{1}{2}[\omega_2^2|W_2|^2 + \omega_1^2|W_3|^2 + 2\omega_1\omega_2 W_2\overline{W}_3] \\
&\quad + \Re\,\frac{1}{2}[\omega_3^2|W_3|^2 + \omega_2^2|W_4|^2 + 2\omega_2\omega_3 W_3\overline{W}_4] \\
&\quad + \Re\,\frac{1}{2}[\omega_3^2|W_2|^2 + \omega_1^2|W_4|^2 + 2\omega_1\omega_3 W_2\overline{W}_4] + \frac{5}{6}|W_5|^2 \\
&= \frac{1}{6}\omega_1^2|W_2|^2 + \frac{|W_2|^2}{2} + \frac{1}{6}\omega_2^2|W_3|^2 + \frac{|W_3|^2}{2} + \frac{1}{6}\omega_3^2|W_4|^2 + \frac{|W_4|^2}{2} \\
&\quad + \Re\left\{\frac{1}{3}\omega_1\omega_2 W_2\overline{W}_3 + \frac{1}{3}\omega_2\omega_3 W_3\overline{W}_4 + \frac{1}{3}\omega_3\omega_1 W_4\overline{W}_2\right\} + \frac{5}{6}|W_5|^2 \\
&\geq \frac{1}{2}\sum_{k=2}^{5}|W_k|^2 + \Re\frac{1}{6}[\omega_1^2|W_2|^2 + \omega_2^2|W_3|^2 + \omega_3^2|W_4|^2 \\
&\quad + 2\omega_1\omega_2 W_2\overline{W}_3 + 2\omega_2\omega_3 W_3\overline{W}_4 + 2\omega_3\omega_1 W_4\overline{W}_2] \\
&= \frac{1}{2}\sum_{k=2}^{5}|W_k|^2 + \frac{1}{6}\,|\omega_1 W_2 + \omega_2 W_3 + \omega_3 W_4|^2 \\
&\geq \frac{1}{2}\sum_{k=2}^{5}|W_k|^2.
\end{aligned}
$$

Thus now we have, for any $\alpha > 0$,

$$
\begin{aligned}
&\alpha\Re\,\langle\overline{R}_{11}(\omega)V_{11}(\omega)W_I, W_I\rangle + |V_{21}(\omega)W_I|^2 \\
&\quad \geq \alpha\left(c_2|W_1|^2 - C_2\sum_{k=2}^{5}|W_k|^2\right) + c_3\sum_{k=2}^{5}|W_k|^2.
\end{aligned}
\tag{3.235}
$$

Take $\alpha = \frac{c_3}{2C_2}$. Then

$$
\Re\langle U_{11}(\omega)W_I, W_I\rangle \geq c|W_I|^2 \quad \text{for some} \quad c > 0.
\tag{3.236}
$$

This completes the proof of Lemma 3.11.1.

COROLLARY *Let $R(\omega)$ be as in (3.226). Then there exists $\beta > 0$ such that $\beta R(\omega)$ is a compensating function for the system of 13 moments.*

Proof. By definition,

$$\bar{L} = \{(Le_k, e_\ell)\}_{1 \leq k, \ell \leq 13}. \tag{3.237}$$

Thus $\bar{L}$ decomposes as

$$\bar{L} = \begin{bmatrix} 0^{5\times 5} & 0^{5\times 8} \\ 0^{8\times 5} & \tilde{L}_{22}^{8\times 8} \end{bmatrix} \tag{3.238}$$

where $\tilde{L}_{22}$ is symmetric positive definite.

Next we exhibit a compensating function for the Boltzmann Equation. Write

$$R(\omega) = \{r_{ij}(\omega)\}_{i,j=1}^{13}. \tag{3.239}$$

Given $\omega \in S^2$, let

$$S(\omega)f \equiv \sum_{k,\ell=1}^{13} \beta r_{k\ell}(\omega)(f, e_\ell)e_k \text{ for some } \beta > 0, \ f \in L^2(v). \tag{3.240}$$

LEMMA 3.11.2 *There exists $\beta > 0$ such that $S(\omega)$ is a compensating function for the Boltzmann equation*

$$(BE) \qquad f_t + v \cdot \nabla_x f + L[f] = \Gamma(f, f).$$

Moreover, $S(\omega) : L^2(v) \to \tilde{W}$.

Proof. The last assertion is clear. Recall that

$$R(\omega) = \sum_{j=1}^{3} \omega_j R^j \tag{3.241}$$

where each R^j is a constant 13×13 real skew–symmetric matrix. Thus $S(\cdot)$ is $C^\infty(S^2)$ and $S(-\omega) = -S(\omega)$, $|\omega| = 1$. This is property (i) of the definition.

Let $f, g \in L^2(v)$. Then

$$(S(\omega)f, g) = \sum_{k,\ell=1}^{13} \beta r_{k\ell}(\omega)(f, e_\ell)\overline{(g, e_k)}. \tag{3.242}$$

Write

$$W = \{W_k\}_{k=1}^{13} = \{(f, e_k)\}_{k=1}^{13}; \ u = \{u_k\}_{k=1}^{13} = \{(g, e_k)\}_{k=1}^{13}. \tag{3.243}$$

Then

$$(S(\omega)f,g) = \beta\langle R(\omega)W,u\rangle. \tag{3.244}$$

Therefore

$$(iS(\omega)f,g) = \beta\langle iR(\omega)W,u\rangle. \tag{3.245}$$

Since $R(\omega)$ is skew-symmetric, $iS(\omega)$ is self-adjoint. This is property (ii). For property (iii): let $f \in L^2_1(v), \omega \in S^2$. Then

$$(S(\omega)(v\cdot\omega)f,f) = \sum_{k,\ell=1}^{13} \beta r_{k\ell}(\omega)((v\cdot\omega)f,e_\ell)\overline{(f,e_k)}. \tag{3.246}$$

Put $f = Pf+(I-P)f$, where P is the orthogonal projection from $L^2(v) \to \tilde{W}$, i.e.,

$$Pf = \sum_{k=1}^{13}(f,e_k)e_k. \tag{3.247}$$

Then, for $W_j = (f,e_j),\ 1 \le j \le 13$,

$$\begin{aligned}((v\cdot\omega)f,e_\ell) &= (Pf+(I-P)f,(v\cdot\omega)e_\ell) \\ &= (Pf,(v\cdot\omega)e_\ell) + ((I-P)f,(v\cdot\omega)e_\ell) \\ &= \left(\sum_{j=1}^{13} W_je_j,(v\cdot\omega)e_\ell\right) + ((I-P)f,(v\cdot\omega)e_\ell) \\ &= \sum_{j=1}^{13} W_j(e_j,(v\cdot\omega)e_\ell) + ((I-P)f,(v\cdot\omega)e_\ell) \\ &= \sum_{j=1}^{13} W_j \sum_{p=1}^{3}(e_j,\omega_pv_pe_\ell) + ((I-P)f,(v\cdot\omega)e_\ell).\end{aligned} \tag{3.248}$$

The first term I_1 equals

$$\begin{aligned}\sum_{j=1}^{13} W_j \sum_{p=1}^{3}\omega_p(e_j,v_pe_\ell) &= \sum_{j=1}^{13} W_j \sum_{p=1}^{3}\omega_p(V^p)_{j\ell} \\ &= \sum_{j=1}^{13} W_j(V(\omega))_{j\ell} = \sum_{j=1}^{13} W_j(V(\omega))_{\ell j}.\end{aligned} \tag{3.249}$$

Therefore

$$\Re(S(\omega)(v\cdot\omega)f,f) = \Re\beta\sum_{k,\ell=1}^{13} r_{k\ell}(\omega)\sum_{j=1}^{13} V_{\ell j}(\omega)W_j\overline{W}_k$$

$$
\begin{aligned}
&+\Re\beta\sum_{k,\ell=1}^{13} r_{k\ell}(\omega)((I-P)f,(v\cdot\omega)e_\ell)\overline{W}_k\\
&=\Re\beta\langle R(\omega)V(\omega)W,W\rangle\\
&+\beta\Re\sum_{k,\ell=1}^{13} r_{k\ell}(\omega)((I-P)f,(v\cdot\omega)e_\ell)\overline{(f,e_k)}\\
&\geq\beta[c_1\|P_0f\|^2-C_1\|(I-P_0)f\|^2]\\
&+\beta\Re\sum_{k,\ell=1}^{13} r_{k\ell}(\omega)\,((I-P)f,(v\cdot\omega)e_\ell)\,\overline{(f,e_k)}
\end{aligned}
$$

where P_0 is the orthogonal projection onto $N(L)$.

Now we claim that the second term J satisfies

$$
|J|\leq c\beta\|f\|\;\|(I-P_0)f\|. \tag{3.250}
$$

Indeed

$$
|(f,e_k)|\leq 1\cdot\|f\| \tag{3.251}
$$

and

(3.252)

$$
|((I-P)f,(v\cdot\omega)e_\ell)|\leq\|(I-P)f\|\;\|(v\cdot\omega)e_\ell\|\leq c\|(I-P)f\|\leq c\|(I-P_0)f\|.
$$

Therefore

$$
\begin{aligned}
\Re\,(S(\omega)(v\cdot\omega)f,f)&\geq\beta c_1\|P_0f\|^2-\beta C_1\|(I-P_0)f\|^2-c\beta\|f\|\;\|(I-P_0)f\|\\
&\geq\beta(c_1-\epsilon)\|P_0f\|^2-\beta C_\epsilon\|(I-P_0)f\|^2
\end{aligned}
$$

for any $\epsilon>0$. Recall that

$$
(L[f],f)\geq\delta_1\|(I-P_0)f\|^2\quad\text{for some}\quad\delta_1>0. \tag{3.253}
$$

Now we add these relations, taking $\epsilon=c_1/2$ and $\beta C_\epsilon=\delta_1/2$. Then

$$
\Re\,(S(\omega)(v\cdot\omega)f,f)+(L[f],f)\geq\delta_2\|f\|^2 \tag{3.254}
$$

for some $\delta_2>0$. This proves the lemma.

3.12. Time Decay Estimates

Consider solutions to the linear equation

$$
f_t+v\cdot\nabla_x f+L[f]=g,\quad f(0,x,v)=f_0(x,v). \tag{3.255}
$$

Now we define

$$\mathcal{H}^\ell = L^2(v, H^\ell(x)), \quad \text{with norm} \quad |\!|\!| f |\!|\!|_\ell = \left(\int \|f(\cdot, v)\|_\ell^2 \, dv\right)^{1/2},$$

$$\mathcal{H}_\beta^\ell = L_\beta^2(v, H^\ell(x)), \quad \text{with norm} \quad |\!|\!| f |\!|\!|_{\mathcal{H}_\beta^\ell} = \left(\int \langle v\rangle^{2\beta} \|f(\cdot, v)\|_\ell^2 \, dv\right)^{1/2},$$

$$L^{p,2} = L^2(v, L^p(x)), \quad \text{with norm} \quad |\!|\!| [f] |\!|\!|_p = \left(\int |f(\cdot, v)|_p^2 \, dv\right)^{1/2}.$$

THEOREM 3.12.1 *Let $\ell \geq 0$. Assume that*

(i) $f_0 \in \mathcal{H}^\ell \cap L^{1,2}$,

(ii) $g \in C^0\left([0, \infty); \mathcal{H}^\ell \cap L^{1,2}\right)$

(iii) $(P_0 g)(t, x, v) = 0$ *for all* $(t, x, v) \in [0, \infty) \times \mathbb{R}^3 \times \mathbb{R}^3$.

Then if $f \in C^0\left([0, \infty); \mathcal{H}_1^\ell\right) \cap C^1\left([0, \infty); \mathcal{H}^{\ell-1}\right)$ *is a solution of (3.255), we have*

$$\begin{aligned} (3.256) \quad |\!|\!| f(t) |\!|\!|_\ell^2 \leq{} & c(1+t)^{-3/2}(|\!|\!| f_0 |\!|\!|_\ell + |\!|\!| [f_0] |\!|\!|_1)^2 \\ & + c\int_0^t (1+t-\tau)^{-3/2}(|\!|\!| g(\tau) |\!|\!|_\ell + |\!|\!| [g(\tau)] |\!|\!|_1)^2 \, d\tau. \end{aligned}$$

Proof. Let $\omega = \frac{\xi}{|\xi|}$ and take the Fourier Transform in x:

$$(3.257) \qquad \hat{f}_t + i|\xi|(v \cdot \omega)\hat{f} + L[\hat{f}] = \hat{g}, \quad \hat{f}(0, \xi, v) = \hat{f}_0(\xi, v).$$

Let $S(\omega)$ be a compensating function as above; $\kappa > 0$. Consider

$$(3.258) \qquad E[\hat{f}](t, \xi) \equiv \|\hat{f}(t, \xi, \cdot)\|^2 - \kappa\tilde{\rho}(\xi)(iS(\omega)\hat{f}(t, \xi, \cdot), \hat{f}(t, \xi, \cdot))$$

where the norm and inner product are over $L^2(v)$ and

$$(3.259) \qquad \tilde{\rho}(\xi) = \frac{|\xi|}{1 + |\xi|^2}.$$

LEMMA 3.12.1 *There exist $\kappa > 0$, $\delta > 0$ such that*

$$(3.260) \qquad \begin{aligned} \frac{1}{2}\|\hat{f}\|^2 \leq E[\hat{f}] \leq 2\|\hat{f}\|^2 \\ \partial_t E[\hat{f}] + \delta\hat{\rho}(\xi)E[\hat{f}] \leq c\|\hat{g}\|^2 \end{aligned}$$

where

$$(3.261) \qquad \hat{\rho}(\xi) = \frac{|\xi|^2}{1 + |\xi|^2}.$$

Assume the lemma for now. Multiply the second statement in the lemma by $e^{t\delta\hat{\rho}(\xi)}$; we get

$$E[\hat{f}] \le e^{-\delta t\hat{\rho}(\xi)}E[\hat{f}_0] + c\int_0^t e^{-\delta\hat{\rho}(\xi)(t-\tau)}\|\hat{g}(\tau,\xi)\|^2\,d\tau. \tag{3.262}$$

Therefore by the first inequality,

$$\|\hat{f}(t,\xi)\|^2 \le ce^{-\delta t\hat{\rho}(\xi)}\|\hat{f}_0(\xi)\|^2 + c\int_0^t e^{-\delta\hat{\rho}(\xi)(t-\tau)}\|\hat{g}(\tau,\xi)\|^2\,d\tau. \tag{3.263}$$

Multiply this by $\langle\xi\rangle^{2\ell} = (1+|\xi|^2)^\ell$ and integrate in ξ:

$$\begin{aligned}\int(1+|\xi|^2)^\ell\|\hat{f}(t,\xi)\|^2\,d\xi &\le c\int(1+|\xi|^2)^\ell e^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}\|\hat{f}_0(\xi)\|^2\,d\xi\\ &\quad + c\int_0^t\int(1+|\xi|^2)^\ell e^{-\delta\frac{|\xi|^2}{1+|\xi|^2}(t-\tau)}\|\hat{g}(\tau,\xi)\|^2\,d\xi\,d\tau.\end{aligned} \tag{3.264}$$

The left side here is $|\!|\!|f(t)|\!|\!|_\ell^2$. Let

$$\begin{aligned}I_0 &= \int(1+|\xi|^2)^\ell e^{-\delta t\frac{|\xi|^2}{1+|\xi|^2}}\|\hat{f}_0(\xi)\|^2 d\xi\\ &= \int_{|\xi|<\epsilon} + \int_{|\xi|>\epsilon} \equiv I_0' + I_0''.\end{aligned} \tag{3.265}$$

Notice that the function

$$h(r) \equiv \frac{r}{1+r} \quad (r \ge 0) \tag{3.266}$$

is increasing. Thus

$$\begin{aligned}I_0'' &\le e^{-\delta t\frac{\epsilon^2}{1+\epsilon^2}}|\!|\!|f_0|\!|\!|_\ell^2 \quad \text{and}\\ I_0' &\le \sup_{|\xi|<\epsilon}\|\hat{f}_0(\xi,\cdot)\|_{L^2(v)}^2\cdot(1+\epsilon^2)^\ell\int_{|\xi|<\epsilon} e^{-\frac{\delta t|\xi|^2}{1+|\xi|^2}}\,d\xi\\ &\le (1+\epsilon^2)^\ell\sup_{|\xi|<\epsilon}\|\hat{f}_0(\xi,\cdot)\|_{L^2(v)}^2\int_{|\xi|<\epsilon} e^{-\frac{\delta t|\xi|^2}{1+\epsilon^2}}\,d\xi.\end{aligned} \tag{3.267}$$

The integral is dominated by

$$\int_{\mathbb{R}^3} e^{-\frac{\delta t|\xi|^2}{1+\epsilon^2}}\,d\xi = \text{const. }(\delta t)^{-3/2}(1+\epsilon^2)^{3/2}.$$

Now we take $\epsilon = 1$ and $t \ge 1$ (without loss of generality) to get

$$\begin{aligned}I_0 &\le e^{-\frac{\delta t}{2}}|\!|\!|f_0|\!|\!|_\ell^2 + c(1+t)^{-3/2}\sup_{|\xi|<1}\|\hat{f}_0(\xi,\cdot)\|_{L^2(v)}^2\\ &\le e^{-\frac{\delta t}{2}}|\!|\!|f_0|\!|\!|_\ell^2 + c(1+t)^{-3/2}\|f_0\|_{L^2(v,L^1(x))}^2\\ &\le c(1+t)^{-3/2}\left(|\!|\!|f_0|\!|\!|_\ell^2 + \|f_0\|_{L^2(v,L^1(x))}^2\right).\end{aligned} \tag{3.268}$$

Clearly the same kind of estimate holds for the g–term in (3.264).

Proof of Lemma 3.12.1. That

$$\frac{1}{2}\|\hat{f}\|^2 \leq E[\hat{f}] \leq 2\|\hat{f}\|^2 \tag{3.269}$$

holds is clear for small κ, say $0 < \kappa \leq \kappa_1$, because

$$\tilde{\rho}(\xi) = \frac{|\xi|}{1+|\xi|^2} \leq \frac{1}{2} \tag{3.270}$$

and $S(\omega)$ is a bounded operator by definition:

$$S(\omega)f = \sum_{k,\ell=1}^{13} \beta r_{k\ell}(\omega)(f, e_\ell)e_k, \quad f \in L^2(v). \tag{3.271}$$

For the differential inequality in the lemma, multiply

$$\hat{f}_t + i|\xi|(v\cdot\omega)\hat{f} + L[\hat{f}] = \hat{g} \tag{3.272}$$

by the conjugate of $\hat{f}$ and take the real part to get

$$\frac{1}{2}\partial_t\|\hat{f}\|^2 + (L[\hat{f}], \hat{f}) = \Re(\hat{g}, \hat{f}). \tag{3.273}$$

Next apply $-i|\xi|S(\omega)$ to the $\hat{f}$ equation:

$$-i|\xi|S(\omega)\hat{f}_t + |\xi|^2 S(\omega)((v\cdot\omega)\hat{f}) - i|\xi|S(\omega)L[\hat{f}] = -i|\xi|S(\omega)\hat{g}. \tag{3.274}$$

Take the inner product with $\hat{f}$ and the real part of the result:

$$\begin{aligned}\Re(-i|\xi|S(\omega)\hat{f}_t, \hat{f}) &+ |\xi|^2\Re\,(S(\omega)(v\cdot\omega)\hat{f}, \hat{f}) \\ &= |\xi|\Re\left\{(iS(\omega)L[\hat{f}], \hat{f}) - (iS(\omega)\hat{g}, \hat{f})\right\}.\end{aligned} \tag{3.275}$$

Since $iS(\omega)$ is self–adjoint, the first term is

$$-\frac{1}{2}\frac{\partial}{\partial t}|\xi|(iS(\omega)\hat{f}, \hat{f}). \tag{3.276}$$

Now we form the following combination: compute $(1+|\xi|^2)$ times (3.273), and add the result to κ times (3.275). The result is

$$\begin{aligned}\partial_t &\left[\frac{(1+|\xi|^2)}{2}\|\hat{f}\|^2 - \frac{\kappa|\xi|}{2}(iS(\omega)\hat{f}, \hat{f})\right] \\ &+ (1+|\xi|^2)(L[\hat{f}], \hat{f}) + \kappa|\xi|^2\Re(S(\omega)(v\cdot\omega)\hat{f}, \hat{f}) \\ &= (1+|\xi|^2)\Re(\hat{g}, \hat{f}) + \kappa|\xi|\Re\{(iS(\omega)L[\hat{f}], \hat{f}) - (iS(\omega)\hat{g}, \hat{f})\},\end{aligned} \tag{3.277}$$

or

$$\frac{\partial}{\partial t}\left\{\frac{(1+|\xi|^2)}{2}E[\hat f]\right\}+(1+(1-\kappa)|\xi|^2)(L[\hat f],\hat f) \tag{3.278}$$
$$+\kappa|\xi|^2\left\{\Re(S(\omega)(v\cdot\omega)\hat f,\hat f)+(L[\hat f],\hat f)\right\}$$
$$=(1+|\xi|^2)\Re(\hat g,\hat f)+\kappa|\xi|\Re\{(iS(\omega)L[\hat f],\hat f)-(iS(\omega)\hat g,\hat f)\}.$$

For $0<\kappa\le 1$, the second term on the left of (3.278) is at least

$$(1-\kappa)(1+|\xi|^2)(L[\hat f],\hat f)\ge(1-\kappa)(1+|\xi|^2)\cdot\delta_1\|(I-P_0)\hat f\|^2. \tag{3.279}$$

By the previous lemma, the third term on the left of (3.278) is at least as large as

$$\kappa|\xi|^2\cdot\delta_2\|\hat f\|^2. \tag{3.280}$$

Therefore the left side of (3.278) is bounded below by

$$\frac{\partial}{\partial t}\left\{\frac{(1+|\xi|^2)}{2}E[\hat f]\right\}+(1-\kappa)\delta_1(1+|\xi|^2)\|(I-P_0)\hat f\|^2+\kappa|\xi|^2\delta_2\|\hat f\|^2. \tag{3.281}$$

Now, to estimate the right side of (3.278): by hypothesis, $P_0 g=0$ and

$$(S(\omega)f,g)=\sum_{k,\ell=1}^{13}\beta r_{k\ell}(\omega)(f,e_\ell)\overline{(g,e_k)}. \tag{3.282}$$

Therefore the absolute value of the first term on the right of (3.278) equals

$$|(1+|\xi|^2)((I-P_0)\hat g,\hat f)|\le(1+|\xi|^2)\|\hat g\|\ \|(I-P_0)\hat f\|. \tag{3.283}$$

Now use

$$S(\omega)L[f]=\beta\sum_{k,\ell}r_{k\ell}(\omega)(L[f],e_\ell)e_k=\beta\sum_{k,\ell}r_{k\ell}(\omega)(f,L[e_\ell])e_k \tag{3.284}$$

and

$$|(f,L[e_\ell])|\le\|(I-P_0)f\|. \tag{3.285}$$

Thus the absolute value of the second term is dominated by

$$c\kappa|\xi|\left(|(iS(\omega)L[\hat f],\hat f)|+|(iS(\omega)\hat g,\hat f)|\right) \tag{3.286}$$
$$\le c\kappa|\xi|\left(\|(I-P_0)\hat f\|\ \|\hat f\|+\|\hat g\|\ \|\hat f\|\right).$$

Thus for any $\epsilon > 0$, the right side of (3.278) is not more than

$$\begin{aligned}&\epsilon(1+|\xi|^2)\|(I-P_0)\hat{f}\|^2 + c_\epsilon(1+|\xi|^2)\|\hat{g}\|^2\\ &\quad + \frac{\kappa\epsilon}{2}|\xi|^2\|\hat{f}\|^2 + \kappa c_\epsilon\|(I-P_0)\hat{f}\|^2 + \frac{\kappa\epsilon}{2}|\xi|^2\|\hat{f}\|^2 + c_\epsilon\|\hat{g}\|^2\\ &\le (\epsilon+\kappa c_\epsilon)(1+|\xi|^2)\|(I-P_0)\hat{f}\|^2 + \kappa\epsilon\|\hat{f}\|^2|\xi|^2 + c_\epsilon(1+|\xi|^2)\|\hat{g}\|^2.\end{aligned}$$

Now take ϵ, κ small such that

$$\epsilon = \min\left\{\frac{\delta_2}{2}, \frac{\delta_1}{6}\right\}, \text{ and then } 0 < \kappa \le \kappa_2, \tag{3.287}$$

where

$$\kappa_2 = \min\left\{\frac{1}{6}, \frac{\delta_1}{6c_\epsilon}\right\}. \tag{3.288}$$

This completes the proof of Lemma 3.12.1.

3.13. Time Decay in Other Norms

Write the Boltzmann equation (BE) as

$$f_t + Bf = \Gamma(f,f)$$

where

$$Bf = v\cdot\nabla_x f + L[f].$$

Thus formally

$$f(t) = e^{-tB}f_0 + \int_0^t e^{-(t-\tau)B}\Gamma(f,f)(\tau)\,d\tau. \tag{3.289}$$

To validate this we need to check that $-B$ generates a strongly continuous semigroup.

LEMMA 3.13.1 *$-B$ generates a strongly continuous semigroup e^{-tB} in $\mathcal{H}^\ell$, $\ell \ge 0$, and in $\dot{B}^\ell_\beta$ for $\ell \ge 0$, $\beta > \frac{3}{2}$.*

Proof. We sketch the proof for the choice $\mathcal{H}^\ell$, $\ell \ge 0$. After applying the Fourier transformation in x to the equation

$$f_t + v\cdot\nabla_x f + L[f] = g, \quad \text{or} \quad f_t + Bf = g$$

we get

$$\hat{f}_t + i\xi\cdot v\hat{f} + L[\hat{f}] = \hat{g}.$$

According to the Hille–Yosida theorem, we want a bound on $(B+\lambda I)^{-1}$ in the appropriate norm. Thus we want to estimate a solution to the equation

$$\lambda\hat{f} + i\xi\cdot v\hat{f} + L[\hat{f}] = \hat{g}.$$

Write $\rho_\ell(\xi) = (1+|\xi|^2)^\ell$. We multiply this equation by the conjugate of $\rho_\ell(\xi)\hat{f}$, integrate over $\mathbb{R}^3_\xi \times \mathbb{R}^3_v$ and take the real part of the result to get

$$\Re\lambda \int \rho_\ell |\hat{f}|^2 \, d\xi \, dv + \Re \int \rho_\ell \overline{\hat{f}} \, L[\hat{f}] \, d\xi \, dv = \Re \int \rho_\ell \overline{\hat{f}} \, \hat{g} \, d\xi \, dv.$$

For $\Re\lambda > 0$ we then get, with the help of the Schwarz inequality and properties of L

$$\Re\lambda \int \rho_\ell |\hat{f}|^2 \, d\xi \, dv \leq \left(\int \rho_\ell |\hat{f}|^2 \, d\xi \, dv \right)^{\frac{1}{2}} \cdot \left(\int \rho_\ell |\hat{g}|^2 \, d\xi \, dv \right)^{\frac{1}{2}},$$

i.e.,

$$\left(\int \rho_\ell |\hat{f}|^2 \, d\xi \, dv \right)^{\frac{1}{2}} \leq \frac{1}{\Re\lambda} \left(\int \rho_\ell |\hat{g}|^2 \, d\xi \, dv \right)^{\frac{1}{2}}.$$

This says

$$\|(B+\lambda I)^{-1} g\| \leq \frac{1}{\Re\lambda} \|g\|, \quad \text{or}$$

$$\|(B+\lambda I)^{-1}\| \leq \frac{1}{\Re\lambda}.$$

This is the required estimate of the Hille–Yosida theorem. We have previously shown that for $\beta \geq 1$ we have $L^2_\beta \subseteq D(L)$, in fact, for such β we have $\nu(v)f$, $Kf \in L^2$. This set is dense and we are done.

LEMMA 3.13.2 *(i) Let*

$$\ell \geq 0, \ \beta > 3/2, \ f_0 \in \dot{B}^\ell_\beta \cap L^2(v, L^1(x)). \tag{3.290}$$

Then

$$|\!|\!| e^{-tB} f_0 |\!|\!|_{\ell,\beta} \leq c(1+t)^{-3/4} (|\!|\!| f_0 |\!|\!|_{\ell,\beta} + \|f_0\|_{L^2(v,L^1(x))}). \tag{3.291}$$

(ii) Let

$$\ell > 3/2, \ \beta > 5/2, \ h_i \in C^0\left([0,\infty); \dot{B}^\ell_\beta\right), \quad i = 1,2. \tag{3.292}$$

Put

$$\tilde{f}(t) = \int_0^t e^{-(t-\tau)B} \Gamma(h_1,h_2)(\tau) \, d\tau. \tag{3.293}$$

Then

$$\tilde{f} \in C^0\left([0,\infty); \dot{B}^\ell_\beta\right) \tag{3.294}$$

and

$$|\!|\!| \tilde{f} |\!|\!|_{\ell,\beta} \leq c(1+t)^{-3/4} |\!|\!| h_1 |\!|\!|_{\ell,\beta,t} |\!|\!| h_2 |\!|\!|_{\ell,\beta,t} \tag{3.295}$$

where

$$\|\!|h|\!\|_{\ell,\beta,t} = \sup_{0\le\tau\le t} (1+\tau)^{3/4} \|\!|h(\tau)|\!\|_{\ell,\beta}. \tag{3.296}$$

Proof. We first prove (i) under the additional assumption that f_0 has compact support in v. Let $f(t) = e^{-tB} f_0$. This is a solution of the linear equation

$$f_t + Bf = 0 \quad \text{with data} \quad f_0. \tag{3.297}$$

Thus

$$f_0 \in \dot{B}^\ell_\alpha \cap L^2(v, L^1(x)) \quad \text{for all} \quad \alpha \ge \beta \tag{3.298}$$

by the additional assumption. Therefore

$$f \in C^0\left([0,\infty);\, \dot{B}^\ell_\alpha\right) \cap C^1\left([0,\infty);\, \dot{B}^{\ell-1}_{\alpha-1}\right) \quad \text{for all} \quad \alpha \ge \beta. \tag{3.299}$$

Now we can apply the previous estimate:

$$\|\!|f(t)|\!\|_\ell \le c(1+t)^{-3/4} \left(\|\!|f_0|\!\|_\ell + \|f_0\|_{L^2(v,L^1(x))}\right). \tag{3.300}$$

Next, rewrite the linearized Boltzmann equation:

$$f_t + v\cdot\nabla_x f + L[f] = 0 \quad \text{as} \quad f_t + v\cdot\nabla_x f + \nu(v) f = Kf \tag{3.301}$$

and consider

$$\begin{aligned} &\frac{d}{ds}[e^{\nu(v)s} f(s, x - v(t-s), v)] \\ &= \nu(v) e^{\nu(v)s} f + e^{\nu(v)s}(f_t + v\cdot\nabla_x f)(s, x - v(t-s), v) \\ &= e^{\nu(v)s} Kf(s, x - v(t-s), v). \end{aligned} \tag{3.302}$$

Hence

$$\begin{aligned} f(t,x,v) &= e^{-\nu(v)t} f_0(x - vt, v) \\ &\quad + \int_0^t e^{-\nu(v)(t-s)} Kf(s, x - v(t-s), v)\, ds. \end{aligned} \tag{3.303}$$

It follows from (3.101) and Lemma 3.3.1 that

$$\|\!|Kf|\!\|_{\ell,0} \le c\|\!|f|\!\|_\ell, \quad \|\!|Kf|\!\|_{\ell,m} \le c\|\!|f|\!\|_{\ell,m-1} \qquad (m \ge 1). \tag{3.304}$$

Therefore there exists c, $0 < c \le \nu_0$ such that

$$\|\!|f(t)|\!\|_{\ell,0} \le e^{-ct}\|\!|f_0|\!\|_{\ell,0} + c_1 \int_0^t e^{-c(t-s)} \|\!|f(s)|\!\|_\ell\, ds \tag{3.305}$$

and

(3.306)

$$|||f(t)|||_{\ell,m} \leq e^{-ct}|||f_0|||_{\ell,m} + c_1 \int_0^t e^{-c(t-\tau)}|||f(\tau)|||_{\ell,m-1}\, d\tau \qquad (m = 1, 2, \ldots)$$

Hence

$$\begin{aligned}
|||f(t)|||_{\ell,0} &\leq e^{-ct}|||f_0|||_{\ell,0} + c_1 \int_0^t e^{-c(t-\tau)}|||f(\tau)|||_{\ell}\, d\tau \\
&\leq e^{-ct}|||f_0|||_{\ell,0} + c_3 \int_0^t e^{-c(t-\tau)}(1+\tau)^{-3/4}(|||f_0|||_{\ell} + \|[f_0]\|_1)\, d\tau \\
&\leq c(1+t)^{-3/4}\left[|||f_0|||_{\ell,0} + |||f_0|||_{\ell} + \|[f_0]\|_1\right].
\end{aligned}$$

Since

(3.307)
$$|||f_0|||_{\ell} \leq c|||f_0|||_{\ell,\beta} \qquad (\beta > 3/2)$$

we get

(3.308)
$$|||f(t)|||_{\ell,0} \leq c(1+t)^{-3/4}\left[|||f_0|||_{\ell,\beta} + \|[f_0]\|_1\right].$$

Now iterate in this manner:

$$\begin{aligned}
|||f(t)|||_{\ell,m} &\leq e^{-ct}|||f_0|||_{\ell,m} + c_1 \int_0^t e^{-c(t-\tau)}|||f(\tau)|||_{\ell,m-1}\, d\tau \\
&\leq e^{-ct}|||f_0|||_{\ell,m} + c_1 \int_0^t e^{-c(t-\tau)}[e^{-c\tau}|||f_0|||_{\ell,m-1} \\
&\qquad + c_1 \int_0^\tau e^{-c(\tau-s)}|||f(s)|||_{\ell,m-2}\, ds]\, d\tau \\
&\leq C(1+t)e^{-ct}|||f_0|||_{\ell,m} + C\int_0^t e^{-c(t-s)}|||f(s)|||_{\ell,m-2} \int_s^t d\tau\, ds \\
&= C(1+t)e^{-ct}|||f_0|||_{\ell,m} + C\int_0^t (t-s)e^{-c(t-s)}|||f(s)|||_{\ell,m-2}\, ds,
\end{aligned}$$

etc. Eventually we will have the following: for some $c_0 > 0$

(3.309)
$$|||f(t)|||_{\ell,m} \leq ce^{-c_0 t}|||f_0|||_{\ell,m} + c\int_0^t e^{-c_0(t-s)}|||f(s)|||_{\ell,0}\, ds.$$

Using (3.308) on the right side, we obtain (i) under the additional assumption. Now replace f_0 by $\chi_R f_0$ where $\chi_R = 1$ on $\{|v| \leq R\}$ is a characteristic function. For $f_R(t) = e^{-tB}\chi_R f_0$ we get with a constant independent of R

(3.310)
$$|||f_R(t)|||_{\ell,\beta} \leq c(1+t)^{-3/4}\left(|||f_0|||_{\ell,\beta} + \|f_0\|_{L^2(v,L^1(x))}\right).$$

But

$$f_R(t) \to e^{-tB} f_0 \text{ in } C^0\left([0,\infty);\, \dot{B}^\ell_\beta\right) \text{ as } R \to \infty. \tag{3.311}$$

This proves (i).

For (ii): it suffices to assume as above that h_1, h_2 have compact support in v. So again

$$h_i \in C^0\left([0,\infty);\, \dot{B}^\ell_\alpha\right), \quad i = 1, 2, \quad \text{for all} \quad \alpha \geq \beta. \tag{3.312}$$

Abbreviate $g = \Gamma(h_1, h_2)$. Then since $\alpha - 1 > 3/2$ and $c(1 + |v|) \leq \nu(v)$,

$$\| g \|_\ell \leq c \| g \|_{\ell, \alpha - 1} \leq c \| \nu^{-1} g \|_{\ell, \alpha} \leq c \| h_1 \|_{\ell, \alpha} \| h_2 \|_{\ell, \alpha}. \tag{3.313}$$

Here in the last step we used Theorem 3.8.1.

Next we claim that

$$\| [g] \|_1 \leq c \| h_1 \|_{0,\alpha} \| h_2 \|_{0,\alpha} \text{ for } \alpha > 5/2. \tag{3.314}$$

To get this, we want to bound

$$\| [g] \|_1 = \left(\int |g(\cdot, v)|^2_{L^1(x)}\, dv \right)^{1/2}. \tag{3.315}$$

For the gain term Γ_{gain}, we have using (3.141)

$$\begin{aligned}
\int |\Gamma_{\text{gain}}(h_1, h_2)|\, dx &= \int \left| \int\!\!\int q\mu^{1/2}(u) h_1(u') h_2(v')\, d\omega\, du \right| dx \\
&\leq c\langle v\rangle^{-\alpha} \int\!\!\int\!\!\int q\mu^{1/2}(u) \langle u'\rangle^\alpha |h_1(u')| \langle v'\rangle^\alpha |h_2(v')|\, d\omega\, du\, dx \\
&\leq c(1 + |v|)^{-\alpha} \left(\int\!\!\int\!\!\int q\mu^{1/2}(u) \langle u'\rangle^{2\alpha} h_1^2(u')\, d\omega\, du\, dx \right)^{1/2} \\
&\quad \cdot \left(\int\!\!\int\!\!\int q\mu^{1/2}(u) \langle v'\rangle^{2\alpha} h_2^2(v')\, d\omega\, du\, dx \right)^{1/2} \\
&\leq c(1 + |v|)^{1-\alpha} \| h_1 \|_{0,\alpha} \| h_2 \|_{0,\alpha}.
\end{aligned} \tag{3.316}$$

(Recall that $\int q\, d\omega = 0(|u - v|)$).

Now square and integrate in v: The result follows because

$$\int_{\mathbb{R}^3} (1 + |v|)^{2 - 2\alpha}\, dv < \infty \quad \text{for} \quad \alpha > \frac{5}{2}. \tag{3.317}$$

Since the loss term is easy to handle, we omit its estimate. This proves that

$$\| [g] \|_1 = \| [\Gamma(h_1, h_2)] \|_1 \leq c \| h_1 \|_{0,\alpha} \| h_2 \|_{0,\alpha}. \tag{3.318}$$

Therefore

$$g = \Gamma(h_1, h_2) \in C^0\left([0,\infty);\, \dot{B}^{\ell}_{\alpha-1} \cap L^{1,2}\right) \quad \text{for all} \quad \alpha \geq \beta. \tag{3.319}$$

Hence

$$\tilde{f}(t) \equiv \int_0^t e^{-(t-\tau)B} g(\tau)\, d\tau \tag{3.320}$$

is a solution of $f_t + Bf = g$ with zero data and

$$\tilde{f} \in C^0\left([0,\infty);\, \dot{B}^{\ell}_{\alpha-1}\right) \cap C^1\left([0,\infty);\, \dot{B}^{\ell-1}_{\alpha-2}\right), \quad \text{for all} \quad \alpha \geq \beta. \tag{3.321}$$

Also $P_0 g = 0$ because $P_0\Gamma(h_1, h_2) = 0$. Therefore by the L^2-decay estimate,

$$|||\tilde{f}(t)|||_\ell^2 \leq c \int_0^t (1+t-\tau)^{-3/2}(|||g(\tau)|||_\ell + \|[g(\tau)]\|_1)^2\, d\tau. \tag{3.322}$$

Let

$$\eta(t) = \eta_1(t)\eta_2(t); \quad \eta_i(t) \equiv |||h_i|||_{\ell,\beta,t}. \tag{3.323}$$

Then

$$|||g|||_\ell \leq c|||h_1|||_{\ell,\alpha}|||h_2|||_{\ell,\alpha} \leq c(1+t)^{-3/2}\eta(t) \tag{3.324}$$

and

$$\|[g]\|_1 \leq c|||h_1|||_{0,\alpha}|||h_2|||_{0,\alpha} \leq c(1+t)^{-3/2}\eta(t). \tag{3.325}$$

Therefore

$$\begin{aligned} |||\tilde{f}(t)|||_\ell^2 &\leq c\int_0^t (1+t-\tau)^{-3/2}\eta^2(\tau)(1+\tau)^{-3}\, d\tau \\ &\leq c(1+t)^{-3/2}\eta^2(t). \end{aligned} \tag{3.326}$$

This is the L^2-version of (ii). To strengthen it: write as before the solution $\tilde{f}$ to $\tilde{f}_t + B\tilde{f} = g$ with data $f_0 = 0$ as

$$\tilde{f}(t,x,v) = \int_0^t e^{-\nu(v)(t-\tau)}(K\tilde{f} + g)(\tau, x - v(t-\tau), v)\, d\tau. \tag{3.327}$$

Let

$$G(t,x,v) \equiv \int_0^t e^{-\nu(v)(t-\tau)} g(\tau, x - v(t-\tau), v)\, d\tau. \tag{3.328}$$

We claim that for $0 \le m \le \beta$,

$$\|G(t)\|_{\ell,m} \le c(1+t)^{-3/2}\eta(t). \tag{3.329}$$

Indeed,

$$\begin{aligned}\|G(t,\cdot,v)\|_\ell &\le \int_0^t e^{-\nu(v)(t-\tau)}\|g(\tau,\cdot,v)\|_\ell \, d\tau \\ &\le \nu(v)\sup_{\tau\le t}[\nu^{-1}(v)\|g(\tau,\cdot,v)\|_\ell(1+\tau)^{3/2}]\cdot\int_0^t \frac{e^{-\nu(v)(t-\tau)}}{(1+\tau)^{3/2}}\, d\tau.\end{aligned}$$

Now $\nu(v)$ times the integral here equals

$$\begin{aligned}&\int_0^t \frac{d}{d\tau}e^{-\nu(v)(t-\tau)}\cdot\frac{d\tau}{(1+\tau)^{3/2}} \\ &= \frac{e^{-\nu(v)(t-\tau)}}{(1+\tau)^{3/2}}\Big|_0^t + \frac{3}{2}\int_0^t \frac{e^{-\nu(v)(t-\tau)}}{(1+\tau)^{5/2}}\, d\tau \\ &\le c(1+t)^{-3/2} \text{ uniformly in } v.\end{aligned} \tag{3.330}$$

Therefore

$$\begin{aligned}\langle v\rangle^m\|G(t,\cdot,v)\|_\ell &\le c(1+t)^{-3/2}\sup_{\tau\le t}\left[\langle v\rangle^m\frac{\|g(\tau,\cdot,v)\|_\ell}{\nu(v)}(1+\tau)^{3/2}\right] \\ &\le c(1+t)^{-3/2}\sup_{\tau\le t}\left\{|\!|\!|g\nu^{-1}|\!|\!|_{\ell,m}\cdot(1+\tau)^{3/2}\right\} \\ &\le c(1+t)^{-3/2}\sup_{\tau\le t}\left\{(1+\tau)^{3/4}|\!|\!|h_1|\!|\!|_{\ell,m}\cdot(1+\tau)^{3/4}|\!|\!|h_2|\!|\!|_{\ell,m}\right\} \\ &= c(1+t)^{-3/2}\eta_1(t)\cdot\eta_2(t) \le c(1+t)^{-3/2}\eta(t).\end{aligned}$$

Now recall that

$$\begin{aligned}&|\!|\!|Kf|\!|\!|_{\ell,m} \le c|\!|\!|f|\!|\!|_{\ell,m-1}, \quad \text{and} \\ &|\!|\!|Kf|\!|\!|_{\ell,0} \le c|\!|\!|f|\!|\!|_{\ell}.\end{aligned} \tag{3.331}$$

Therefore for $1 \le m \le \beta$,

$$\begin{aligned}|\!|\!|\tilde f(t)|\!|\!|_{\ell,0} &\le c\int_0^t e^{-c(t-\tau)}|\!|\!|\tilde f(\tau)|\!|\!|_\ell \, d\tau + c(1+t)^{-3/2}\eta(t); \\ |\!|\!|\tilde f(t)|\!|\!|_{\ell,m} &\le c\int_0^t e^{-c(t-\tau)}|\!|\!|\tilde f(\tau)|\!|\!|_{\ell,m-1}\, d\tau + c(1+t)^{-3/2}\eta(t).\end{aligned} \tag{3.332}$$

We iterate this as before to get part (ii) of the lemma.

3.14. The Major Theorem

Finally we can state and prove the main result.

MAJOR THEOREM 3.14.1 *Assume the hard-sphere case. Assume that* $f_0 \in \dot{B}^\ell_\beta \cap L^{1,2}$ *for some* $\ell > 3/2$, $\beta > 5/2$. *If* $\|\!|f_0|\!\|_{\ell,\beta} + \|\,[f_0]\,\|_1$ *is small enough, then the (BE)*

$$f_t + v\cdot\nabla_x f + L[f] = \Gamma(f,f), \quad f(0,x,v) = f_0(x,v) \tag{3.333}$$

has a unique global solution

$$f \in C^0\left([0,\infty);\, \dot{B}^\ell_\beta\right) \cap C^1\left([0,\infty);\, \dot{B}^{\ell-1}_{\beta-1}\right) \tag{3.334}$$

which satisfies

$$\|\!|f(t)|\!\|_{\ell,\beta} \le c(1+t)^{-3/4}(\|\!|f_0|\!\|_{\ell,\beta} + \|\,[f_0]\,\|_1). \tag{3.335}$$

Proof. We show this iteration converges: $f^0 \equiv 0$,

$$f^n(t) = e^{-tB} f_0 + \int_0^t e^{-(t-\tau)B}\Gamma(f^{n-1}, f^{n-1})\, d\tau \qquad (n = 1,2,\ldots) \tag{3.336}$$

Let

$$\Omega f \equiv e^{-tB} f_0 + \int_0^t e^{-(t-\tau)B}\Gamma(f,f)\, d\tau. \tag{3.337}$$

We know that the linear term satisfies

$$\|e^{-tB} f_0\|_{\ell,\beta} \le c(1+t)^{-3/4}\left(\|\!|f_0|\!\|_{\ell,\beta} + \|\,[f_0]\,\|_1\right) \tag{3.338}$$

and that for

$$\tilde{f}(t) \equiv \int_0^t e^{-(t-\tau)B} g(\tau)\, d\tau, \quad \text{with} \quad g = \Gamma(h_1, h_2), \tag{3.339}$$

we have

$$\|\!|\tilde{f}(t)|\!\|_{\ell,\beta} \le c(1+t)^{-3/4} \sup_{\tau\le t}(1+\tau)^{3/4}\|\!|h_1(\tau)|\!\|_{\ell,\beta} \cdot \sup_{\tau\le t}(1+\tau)^{3/4}\|\!|h_2(\tau)|\!\|_{\ell,\beta}.$$

Therefore

$$\begin{aligned}(1+t)^{3/4}\|\!|\Omega f(t)|\!\|_{\ell,\beta} \le c(&\|\!|f_0|\!\|_{\ell,\beta} + \|\,[f_0]\,\|_1) \\ &+ c\left[\sup_{\tau\le t}(1+\tau)^{3/4}\|\!|f(t)|\!\|_{\ell,\beta}\right]^2.\end{aligned} \tag{3.340}$$

For $\delta > 0$ let

$$X_{\ell,\beta} = \left\{ f \in C^0\left([0,\infty);\, \dot{B}^\ell_\beta\right) : \sup_{\tau\le t}(1+\tau)^{3/4}\|\!|f(\tau)|\!\|_{\ell,\beta} \le \delta \right\}. \tag{3.341}$$

Then from (3.340) we have for $f \in X_{\ell,\beta}$

$$\|\Omega f\|_{X_{\ell,\beta}} \leq c_0[\|\!|f_0|\!\|_{\ell,\beta} + \|[f_0]\|_1] + c_1\|f\|^2_{X_{\ell,\beta}}. \tag{3.342}$$

Assume that

$$\|\!|f_0|\!\|_{\ell,\beta} + \|[f_0]\|_1 \leq \frac{\delta}{2c_0}. \tag{3.343}$$

Then

$$\|\Omega f\|_{X_{\ell,\beta}} \leq \frac{\delta}{2} + c_1\delta^2. \tag{3.344}$$

Take $\delta \leq \frac{1}{2c_1}$ to get

$$\|\Omega f\|_{X_{\ell,\beta}} \leq \frac{\delta}{2} + \delta \cdot \frac{1}{2} = \delta. \tag{3.345}$$

Hence

$$\Omega \ : \ X_{\ell,\beta} \to X_{\ell,\beta} \text{ for such } \delta, f_0. \tag{3.346}$$

To see that Ω is a contraction, recall that Γ is bilinear and write

$$\begin{aligned}\Gamma(f_2, f_2) - \Gamma(f_1, f_1) &= \Gamma(f_2 - f_1, f_2) + \Gamma(f_1, f_2)\\ &\quad - \Gamma(f_1, f_1 - f_2) - \Gamma(f_1, f_2)\\ &= \Gamma(f_2 - f_1, f_2) - \Gamma(f_1, f_1 - f_2).\end{aligned} \tag{3.347}$$

Thus

$$\Omega f_2 - \Omega f_1 = \int_0^t e^{-(t-\tau)B}\left[\Gamma(f_2 - f_1, f_2) + \Gamma(f_1, f_2 - f_1)\right] d\tau. \tag{3.348}$$

We apply the above estimates with $h_1 = f_2 - f_1$, $h_2 = f_2$, etc. Thus Ω is a contraction on $X_{\ell,\beta}$ for δ sufficiently small; its unique fixed point is the desired solution.

3.15. The Relativistic Boltzmann Equation

In this section we write down the relativistic Boltzmann equation and state two of the major theorems known regarding asymptotic stability of the equilibrium.

The relativistic Boltzmann equation is

$$V \cdot \nabla_X F = -C(F, F). \tag{$(RB)_0$}$$

Here the dot represents the Lorentz inner product $(+ - - -)$ of 4–vectors,

$$v = (v_1, v_2, v_3),\ V = (v_0, v_1, v_2, v_3),\ X = (x_0, x_1, x_2, x_3),\ x = (x_1, x_2, x_3),$$

$x_0 = -t$ and $\mathcal{C}(F,F)$ is the collision integral. Normalizing the speed of light $c = 1$ and the particle mass $m = 1$, we have $V \cdot V = 1$ or $v_0 = \sqrt{1+|v|^2}$.

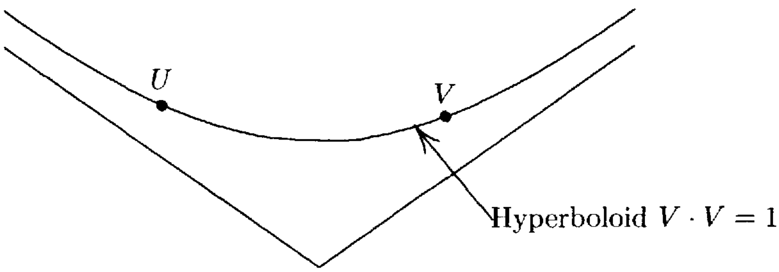

Fig. 3.2 *The Collision Hyperboloid*

It is convenient to separate the time and space variables and to divide $(RB)_0$ by v_0 to obtain

$$(RB) \qquad \partial_t F + \hat{v} \cdot \nabla_x F = Q(F,F)$$

where $Q(F,F) = v_0^{-1}\mathcal{C}(F,F)$ and

$$\hat{v} = \frac{v}{v_0} = \frac{v}{\sqrt{1+|v|^2}}; \tag{3.349}$$

$$Q(F,F)(v) = \frac{1}{2v_0}\iiint \delta(U^2-1)\delta(U'^2-1)\delta(V'^2-1)s\sigma(s,\theta)\cdot$$
$$\delta^{(4)}(U+V-U'-V')\left[F(u')F(v') - F(u)F(v)\right] d^4U d^4U' d^4V \tag{3.350}$$

where $U^2 = U \cdot U = u_0^2 - |u|^2$, $|u|^2 = u_1^2 + u_2^2 + u_3^2$, δ is the delta function in one variable, $\delta^{(4)}$ is the delta function in four variables, and all of the F are evaluated at the same space–time point (t,x). Furthermore $\sigma(s,\theta)$ is called the *differential cross section* or the *scattering kernel*; it is a function of variables s and θ which will be defined below. The delta functions express the conservation of momentum and energy:

$$u' + v' = u + v \tag{3.351}$$
$$\sqrt{1+|u'|^2} + \sqrt{1+|v'|^2} = \sqrt{1+|u|^2} + \sqrt{1+|v|^2}.$$

Of course, the 12-fold integral in (3.350) defining Q may be reduced to a 5-fold integral by carrying out the delta function integrations (see below).

A relativistic Maxwellian is characterized as a particle distribution $\mu(v)$ which minimizes the entropy subject to constant mass, momentum and energy. It is an equilibrium solution of (RB) since $Q(\mu, \mu) = 0$, and it has the form

$$\mu(v) = e^{a+b\cdot v - c\sqrt{1+|v|^2}} \tag{3.352}$$

where $a \in \mathbb{R}$, $b \in \mathbb{R}^3$ and $c \in \mathbb{R}^+$, $c > |b|$ are five parameters (constants).

We consider at first a solution $F(t, x, v)$ of (RB) which has period 2π in each x variable and satisfies an initial condition $F(0, x, v) = F^0(x, v)$. We assume that the initial distribution $F^0(x, v)$ is close to a Maxwellian $\mu(v)$. The 5 parameters a, $b = (b_1, b_2, b_3)$ and c are chosen so that F^0 and μ have the same total mass, energy and momentum:

(3.353)

$$0 = \int\int (F^0 - \mu)\, dx\, dv = \int\int v(F^0 - \mu)\, dx\, dv = \int\int \sqrt{1+|v|^2}(F^0 - \mu)\, dx\, dv$$

where the integration is over $x \in B \equiv (0, 2\pi)^3$ and $v \in \mathbb{R}^3$. See Theorem 3.15.3 below. The periodicity condition in x implies that (RB) has been solved in a box with specular boundary conditions.

As was seen earlier in this chapter in the classical case, the proof of asymptotic stability is based on the fact that the linearized equation possesses some dissipation, due to the increase of the entropy.

In the relativistic case, we write the linearized equation of (RB) as

$$\partial_t f + \hat{v} \cdot \nabla_x f + \nu(v) f + K f = 0 \tag{3.354}$$

where K is a certain integral operator in v, and $\nu(v) > 0$ is a scalar function of v which represents the dissipation. In order to prove dissipation on the operator level in the sense of spectral theory, one needs compactness properties of the solutions of (3.354). Compactness in the v variable follows from the form of K, while compactness in the x variable follows from the fact that v-averages of solutions of transport equations tend to be x-smoothing. The precise condition used is a kind of relative compactness of operators, called A-smoothing. To some extent one can follow the abstract approach that Shizuta [49] applied to the classical Boltzmann equation.

For background on the relativistic equation we mention the book of de Groot et.al [16]. The linearized relativistic equation (3.354) is solved by Dudyński and Ekiel-Jezewska [22].

Below we explicitly write the equation and state the collision invariants and entropy inequality. Then the form of the linearized equation is specified, and the function spaces and the main theorem are formulated. Details may be found in [29].

Let us begin by defining the remaining variables in the collision integral (3.350). We define

$$
\begin{aligned}
s &= (U+V)^2 = (u_0+v_0)^2 - |u+v|^2 \\
&= 2u_0v_0 - 2u\cdot v + u_0^2 - |u|^2 + v_0^2 - |v|^2 \\
&= 2(\sqrt{1+|u|^2}\sqrt{1+|v|^2} - u\cdot v + 1)
\end{aligned} \tag{3.355}
$$

$$
\begin{aligned}
4g^2 &= -(U-V)^2 = -(u_0-v_0)^2 + |u-v|^2 \\
&= 2u_0v_0 - 2u\cdot v - u_0^2 + |u|^2 - v_0^2 + |v|^2 \\
&= 2(\sqrt{1+|u|^2}\sqrt{1+|v|^2} - u\cdot v - 1) \\
&= s-4,
\end{aligned} \tag{3.356}
$$

and

$$
\cos\theta = \frac{(V-U)\cdot(V'-U')}{(V-U)^2}. \tag{3.357}
$$

Furthermore, we define the Møller velocity as the scalar v_M given by

$$
v_M^2 = |\hat{v}-\hat{u}|^2 - |\hat{v}\times\hat{u}|^2 = \frac{s(s-4)}{4v_0^2u_0^2}
$$

or

$$
v_M = \frac{2g\sqrt{1+g^2}}{v_0u_0}. \tag{3.358}
$$

The two expressions for v_M^2 are equal because

$$
\begin{aligned}
\frac{1}{4}s(s-4) = sg^2 &= (u_0v_0 - u\cdot v + 1)(u_0v_0 - u\cdot v - 1) \\
&= |u|^2 + |v|^2 + |u|^2|v|^2 - 2u_0v_0u\cdot v + (u\cdot v)^2 \\
&= u_0^2|v|^2 + v_0^2|u|^2 - 2u_0v_0u\cdot v - |u\times v|^2 \\
&= u_0^2v_0^2\left[\frac{|v|^2}{v_0^2} + \frac{|u|^2}{u_0^2} - 2\frac{u}{u_0}\cdot\frac{v}{v_0} - \left|\frac{u}{u_0}\times\frac{v}{v_0}\right|^2\right].
\end{aligned}
$$

It may not be a priori clear that the expression for $\cos\theta$ is well-defined. We study this is Lemma 3.15.3 at the end of this chapter.

In de Groot et al. [16] the delta function integrations in the collision integral are carried out, resulting in the equation

$$
(RB)\quad \partial_t F + \hat{v}\cdot\nabla_x F = \int_{\mathbb{R}^3}\int_{S^2} v_M\sigma(g,\theta)\cdot[F(u')F(v') - F(u)F(v)]\,d\Omega\,du
$$

where $d\Omega$ is the element of surface area on S^2 and we have written σ as a function of g and θ. The variables u, v, u', v' are related by the equations (3.351). These equations allow u' and v' to be written in terms of u and v and a pair of variables θ and φ which run over the unit sphere S^2. Equation (RB)

is the result of one such representation. A slightly different representation is given in Appendix II of [29]. Indeed, we may write for a vector $\Omega \in S^2$

$$u' = u + a(u, v, \Omega)\Omega, \quad v' = v - a(u, v, \Omega)\Omega. \tag{3.359}$$

With the help of [28], we set

$$e = \sqrt{1 + |u|^2} + \sqrt{1 + |v|^2}$$

and obtain

$$a(u, v, \Omega) = \frac{2e[\Omega \cdot (\hat{v} - \hat{u})]\sqrt{1 + |u|^2}\sqrt{1 + |v|^2}}{e^2 - (\Omega \cdot (v + u))^2}.$$

Another difference from the classical situation is that the Jacobian is no longer unity. Indeed, from [28] we know that

$$\frac{\partial(u', v')}{\partial(u, v)} = -\frac{\sqrt{1 + |u'|^2}\sqrt{1 + |v'|^2}}{\sqrt{1 + |u|^2}\sqrt{1 + |v|^2}}.$$

We remark that in the center-of-mass coordinate frame where $u + v = 0$, $\sqrt{s}$ is the energy, $-2g$ is the relative momentum, and θ is the scattering angle.

In the classical limit, where $|u| + |v| << 1$, we have $s \sim 4 + |u - v|^2$, so that $v_M \sim |u - v|$ and (RB) formally becomes the classical Boltzmann equation.

The collision invariants and the entropy are essentially the same as in the classical case, modulo the form of energy conservation. Define the symmetrized collision operator

$$Q^*(f, g) = \frac{1}{2} \int\int v_M \sigma [f(v')g(u') + f(u')g(v') - f(v)g(u) - f(u)g(v)]\, du\, d\Omega. \tag{3.360}$$

The collision operator in (RB) is $Q(F,F) = Q^*(F,F)$. For example, for f, g smooth and small at infinity, the collision operator satisfies

$$\int Q^*(f, g)\, dv = 0, \quad \int vQ^*(f, g)\, dv = 0, \quad \int \sqrt{1 + |v|^2}\, Q^*(f, g)\, dv = 0,$$
$$\int Q(f, f)(1 + \log f)\, dv \leq 0.$$

Thus it follows that, for the solutions of (RB), the mass $\int F\, dv\, dx$, the momentum $\int vF\, dv\, dx$, and the energy $\int \sqrt{1 + |v|^2}\, F\, dv\, dx$ are invariants. Furthermore $\int F \log F\, dv\, dx$ is a non-increasing function of t.

The linearization proceeds as follows. First of all, we may normalize the Maxwellian to be

$$\mu(v) = e^{-\sqrt{1+|v|^2}}. \tag{3.361}$$

As we did earlier in this chapter, we change variables from F to f by

$$F = \mu + \sqrt{\mu} f. \tag{3.362}$$

Substituting this into (RB), we have

$$\begin{aligned}
\partial_t f + \hat{v} \cdot \nabla_x f &= \frac{1}{\sqrt{\mu}} Q^*(\mu + \sqrt{\mu} f, \mu + \sqrt{\mu} f) \\
&= \frac{2}{\sqrt{\mu}} Q^*(\mu, \sqrt{\mu} f) + \frac{1}{\sqrt{\mu}} Q^*(\sqrt{\mu} f, \sqrt{\mu} f)
\end{aligned}$$

or

$$\partial_t f + \hat{v} \cdot \nabla_x f = -\nu(v) f - K f + \tilde{Q}(f,f) \tag{3.363}$$

where

$$\begin{aligned}
\tilde{Q}(f,f) &= \frac{1}{\sqrt{\mu}} Q(\sqrt{\mu} f, \sqrt{\mu} f) \\
&= \iint \frac{g\sqrt{1+g^2}}{u_0 v_0} \sigma(g,\theta) \sqrt{\mu(u)} \left[f(u') f(v') - f(u) f(v) \right] du\, d\Omega,
\end{aligned} \tag{3.364}$$

$$\nu(v) = \iint \frac{g\sqrt{1+g^2}}{u_0 v_0} \sigma(g,\theta) \mu(u)\, du\, d\Omega, \tag{3.365}$$

and, with σ abbreviating $\sigma(g,\theta)$,

(3.366)

$$K f(v) = \iint \frac{g\sqrt{1+g^2}}{u_0 v_0} \sigma \sqrt{\mu(u)} \Big[-\sqrt{\mu(u')} f(v') - \sqrt{\mu(v')} f(u') + \sqrt{\mu(v)} f(u) \Big] du\, d\Omega.$$

If F is a solution of (RB) and f is defined as in (3.362), then we have

$$0 = \iint \sqrt{\mu} f\, dv\, dx = \iint v \sqrt{\mu} f\, dv\, dx = \iint \sqrt{1+|v|^2} \sqrt{\mu} f\, dv\, dx. \tag{3.367}$$

The entropy implies the following dissipative property of the linearized operator $L \equiv \nu + K$:

$$\int (-\nu - K) f(v) \cdot f(v)\, dv < 0 \tag{3.368}$$

for any function $f(v)$ which satisfies (3.367).

The solution spaces are defined as follows. In the x variable we use the space C^k of periodic functions whose kth derivatives are continuous, or the Sobolev spaces H^k of periodic functions whose kth derivatives belong to L^2.

Let X denote either C^k for $k \geq 0$ or H^k for $k \geq 2$. Let $\rho_\alpha(v) = (1+|v|^2)^{\alpha/2}$. Define $G_\alpha(X)$ as the space of continuous functions $f : \mathbb{R}^3 \to X$ for which

$$\|f\| = \sup_{v \in \mathbb{R}^3} \rho_\alpha(v)|f(v)|_X < \infty.$$

(Earlier in this chapter, when $X = H^\ell$, we called these spaces $B_\alpha^\ell = L_\alpha^\infty(v, H^\ell)$. Since the space X can be chosen differently in this section, we use the different notation of $G_\alpha(X)$ to avoid confusion).

Furthermore define $G_\alpha^0(X)$ as the closed subspace of $G_\alpha(X)$ for which

$$\lim_{|v|\to\infty} \rho_\alpha(v)|f(v)|_X = 0,$$

furnished with the same norm. Let

$$Y = \quad \text{either} \quad G_\alpha^0(H^\ell) \quad \text{or} \quad G_\alpha^0(C^k). \tag{3.369}$$

where $\alpha \geq 0$, $k \geq 0$ and $\ell \geq 2$.

We now state the hypothesis on the collision cross–section $\sigma(g,\theta)$. It is to satisfy

$$c_1 \frac{g^{\beta+1}}{1+g} \sin^\gamma \theta \leq \sigma(g,\theta) \leq c_2(g^\beta + g^{-\delta}) \sin^\gamma \theta \tag{3.370}$$

and

$$\left|\frac{\partial \sigma}{\partial g}\right| \leq c_3(g^{\beta'} + g^{-\delta'}) \sin^{\gamma'} \theta \tag{3.371}$$

where c_1, c_2 and c_3 are positive constants, $0 \leq \delta < \frac{1}{2}$, $0 \leq \beta < 2 - 2\delta$, $0 \leq \delta' < 4$, $\beta' \geq 0, \gamma' > -2$, and either $\gamma \geq 0$ or

$$|\gamma| < \min\left\{2-\beta,\ \frac{1}{2} - \delta,\ \frac{1}{3}(2 - 2\delta - \beta)\right\}. \tag{3.372}$$

Now the main theorem can be stated.

THEOREM 3.15.1 *([29]) Assume that $\sigma(g,\theta)$ satisfies (3.370)-(3.372). Let Y be either of the spaces (3.369), where $\alpha > \frac{1}{2}(3+\beta)$. If $f^0 \in Y$ satisfies (3.367) and $\|f^0\|_Y$ is sufficiently small, then there exists $h > 0$ and a unique global solution $f(t,x,v)$ of (RB) which is periodic in x and satisfies $f(0,x,v) = f^0(x,v)$, $f \in C([0,\infty); Y)$ and*

$$\|f(t)\|_Y \leq c_4 \|f^0\|_Y e^{-ht}, \qquad 0 \leq t < \infty. \tag{3.373}$$

The required estimates on the collision operator are similar to those in the classical setting. Indeed, under the hypothesis (3.370) - (3.372) on σ, we have the following estimate. Consider any of the spaces

$$G_\alpha^0(H^\ell) \text{ or } G_\alpha^0(C^k)$$

for $\alpha \geq \beta/2$, $\ell \geq 2$, $k \geq 0$. Let $\|\cdot\|_\alpha$ be the norm in any one of these spaces. If $f \in G^0_\alpha$, then $\tilde{Q}(f,f) \in G^0_{\alpha-\frac{\beta}{2}}$ and

$$\|\tilde{Q}(f,f)\|_{\alpha-\frac{\beta}{2}} \leq c\|f\|^2_\alpha. \tag{3.374}$$

The map $\tilde{Q}$ is continuous from G^0_α into $G^0_{\alpha-\beta/2}$; in fact

$$\|\tilde{Q}(f,f) - \tilde{Q}(g,g)\|_{\alpha-\frac{\beta}{2}} \leq c(\|f\|_\alpha + \|g\|_\alpha)\|f-g\|_\alpha. \tag{3.375}$$

Next we specify the relevant integral operators. Recall that the equation for the perturbation f (using $F = \mu + \mu^{1/2} f$) can be written as

$$f_t + \hat{v}\cdot\nabla_x f = -\nu(v)f - Kf + \tilde{Q}(f,f) \tag{LRB}$$

where

$$\nu(v) = \int\int \frac{g\sqrt{1+g^2}\sigma(g,\theta)}{u_0 v_0}\mu(u)\,du\,d\Omega \tag{3.376}$$

and (with $\sigma = \sigma(g,\theta)$)

$$Kf = -\int\int \frac{g\sqrt{1+g^2}\,\sigma\sqrt{\mu(u)}}{u_0 v_0}\left[\sqrt{\mu(u')}f(v') + \sqrt{\mu(v')}f(u') - \sqrt{\mu(v)}f(u)\right]du\,d\Omega. \tag{3.377}$$

Here $u_0 = \sqrt{1+|u|^2}$, $v_0 = \sqrt{1+|v|^2}$. We define

$$\ell = \frac{1}{2}(u_0+v_0) \text{ and } j = \frac{|u\times v|}{2g}. \tag{3.378}$$

Then from [16], [22] we know that $Kf = K_2 f - K_1 f$, where $K_{1,2}$ are the integral operators

$$K_i f(t,x,v) = \int_{\mathbb{R}^3} k_i(u,v) f(t,x,u)\,du \quad (i = 1,2) \tag{3.379}$$

with the symmetric kernels

$$k_1(u,v) = c_1\frac{g\sqrt{1+g^2}\,e^{-\ell}}{u_0 v_0}\int_0^\pi \sigma(g,\theta)\sin\theta\,d\theta; \tag{3.380}$$

$$k_2(u,v) = \frac{c_2(1+g^2)^{3/2}}{g u_0 v_0}\int_0^\infty e^{-\ell\sqrt{1+y^2}}\sigma\left(\frac{g}{\sin(\frac{\psi}{2})},\psi\right)\frac{y(1+\sqrt{1+y^2})}{\sqrt{1+y^2}}I_0(jy)\,dy \tag{3.381}$$

where

$$\sin\left(\frac{\psi}{2}\right) = \frac{\sqrt{2}g}{[g^2-1+(g^2+1)\sqrt{1+y^2}]^{1/2}} \tag{3.382}$$

and I_0 is the Bessel function of the second kind of order zero.

In order to estimate the kernels, we use the following elementary inequalities:

$$(i) \qquad \frac{[|u \times v|^2 + |u - v|^2]^{1/2}}{2u_0^{1/2} v_0^{1/2}} \leq g \leq \frac{1}{2}|u - v|$$

$$(ii) \qquad v_M = \frac{2g\sqrt{1+g^2}}{u_0 v_0} \leq 2$$

$$(iii) \qquad \ell^2 - j^2 = \frac{g^2+1}{4g^2}|u - v|^2$$

$$(iv) \qquad \ell^2 - j^2 \geq \max\{g^2 + 1, \frac{1}{4}|u - v|^2\}$$

$$(v) \qquad \frac{1}{\sqrt{2}} g(1 + y^2)^{1/4} \leq \frac{g}{\sin(\frac{\psi}{2})} \leq (g^2 + 1)^{1/2}(1 + y^2)^{1/4}$$

$$(vi) \qquad \cos\frac{\psi}{2} \geq \frac{y}{2\sqrt{1+y^2}}.$$

We establish the bounds on g only. The bound $g \leq \frac{1}{2}|u - v|$ is equivalent to $4g^2 \leq |u - v|^2$, i.e., to $2u_0 v_0 - 2u \cdot v - 2 \leq |u - v|^2$, i.e., to

$$u_0 v_0 \leq u \cdot v + 1 + \frac{1}{2}|u - v|^2 = 1 + \frac{1}{2}(|u|^2 + |v|^2).$$

Squaring both sides, we obtain the upper bound in (i). For the lower bound in (i), we write

$$\begin{aligned} 2g^2 &= \frac{(1 + |u|^2)(1 + |v|^2) - (1 + u \cdot v)^2}{u_0 v_0 + u \cdot v + 1} \\ &\geq \frac{|u|^2 + |v|^2 + |u|^2|v|^2 - 2u \cdot v - (u \cdot v)^2}{2u_0 v_0} \\ &= \frac{|u - v|^2 + |u \times v|^2}{2u_0 v_0}. \end{aligned}$$

From these bounds and the assumptions on σ, one obtains the following estimates:

i) There is a positive constant c, depending only on γ, β such that

$$c^{-1} v_0^{\beta/2} \leq \nu(v) \leq c v_0^{\beta/2}$$

for all $v \in \mathbb{R}^3$, where $v_0 = \sqrt{1 + |v|^2}$,

ii) There is a positive constant c such that

$$k_1(u, v) \leq \frac{c e^{-\frac{1}{2}\ell}}{|u - v|^\delta},$$

iii) There is a positive constant c such that

$$k_2(u,v) \leq \frac{c(1+|v|)^{\frac{3|\gamma|+\beta+2\delta}{2}} \cdot e^{-c|u-v|}}{[|u\times v|^2+|u-v|^2]^{1/2}|u-v|^{\delta+|\gamma|}}.$$

provided $|\gamma|+\beta \leq 2$, $|\gamma|<2$ if $\gamma>0$.

As a Corollary, we see that the kernel k_1 may be taken to satisfy the same bound as k_2. These estimates require careful study of explicit integrals involving Bessel functions.

Lastly we list several integral properties of $k(u,v) = k_2(u,v) - k_1(u,v)$ which can be used as in the classical argument:

i) $k(u,v)$ is a symmetric kernel,
ii) $\sup_v \int |k(u,v)|\, du < \infty$,
iii) $\sup_v \int k^2(u,v)\, du < \infty$,
iv) $\int |k(u,v)|(1+|u|^2)^{-\alpha/2}\, du \leq c(1+|v|^2)^{-\frac{1}{2}(\alpha+\eta)}$ for any $\alpha \geq 0$, where

$$\eta = 1 - \frac{1}{2}[3|\gamma|+\beta+2\delta] > 0.$$

This concludes our brief sketch of the set-up for (RB) near equilibrium in the case of periodic boundary conditions. The proof of smoothing is omitted; we refer to [29] for details.

Now we turn to the study of the pure Cauchy problem for (RB) near a relativistic Maxwellian. Although some of the basic estimates above can be utilized, the method of proof is fundamentally different from the periodic case because smoothing operators in unbounded space lose their compactness and therefore the decay to equilibrium is no longer exponential. One can extend Kawashima's proof (done earlier in this chapter) to the relativistic case, thereby obtaining smooth solutions which tend to equilibrium. Recently Andréasson [1] has also studied the approach to equilibrium of the general weak solutions of (RB). This is analogous to the non-relativistic proof in [42] and is based on a regularizing property of the gain term. Furthermore the nonstandard approach of Arkeryd [3] is put into context for the relativistic situation.

An energy estimate is the most direct way to exploit the increase of entropy. For the linearized equation, we have seen that the entropy increase corresponds to the property $(Lf,f) \geq 0$. However, the entropy is difficult to exploit because of the five-dimensional nullspace of L. After Fourier transforming in the x variable, we have seen in Kawashima's approach the construction of a modified energy form which is positive definite. The modification is accomplished via a compensating function, an operator of finite rank using 13 moments which are related to the streaming term. In the relativistic case one requires 14 moments because the relativistic energy $\sqrt{1+|v|^2}$ depends nonquadratically on the momentum v. The construction of the relativistic compensating function then is coupled to the estimates on L and $\tilde{Q}$ from above to obtain a $t^{-3/4}$ decay rate, first for the linearized problem and then for the nonlinear one.

The hypotheses on the scattering kernel are slightly different in this case. We assume that σ satisfies the following:

$$c_1 \frac{g^{\beta+1}}{1+g} \sin^\gamma \theta \le \sigma(g,\theta) \le c_2(g^\beta + g^{-\delta}) \sin^\gamma \theta \tag{3.383}$$

where c_1, c_2 are positive constants, $0 \le \delta < 1/2$, $0 \le \beta < 2-2\delta$ and either $\gamma \ge 0$ or $|\gamma| < \min\{2-\beta, \frac{1}{2}-\delta, \frac{1}{3}(2-2\delta-\beta)\}$.

We use $\| \ \ \|$ to denote the norm, and $(\ , \ )$ the inner product, in $L^2(\mathbb{R}^3)$. $H^k(\mathbb{R}^3)$ denotes the usual Sobolev space with norm $\| \ \ \|_k$. For functions $f(x,v)$ depending on both variables we take norms first in x and then in v. Thus $L^q(L^p)$ denotes $L^q(\mathbb{R}^3_v; L^p(\mathbb{R}^3_x))$. We use the special notation

$$|||f|||_k = \left\{ \int_{\mathbb{R}^3} \|f(\cdot, v)\|_k^2 \, dv \right\}^{1/2} \tag{3.384}$$

for the norm in $L^2(H^k)$. We will use the same weighted L^∞ spaces $G_\alpha(H^k)$ and $G^0_\alpha(H^k)$ from above.

In terms of these spaces we now state the main result. Recall from (3.362) the definition of f in terms of μ and F.

THEOREM 3.15.2 ([30]) *Write $Lf = \nu(v)f + Kf$ and consider the Cauchy Problem*

$$(RB') \qquad \partial_t f + \hat{v} \cdot \nabla_x f + Lf = \tilde{Q}(f,f)$$

$$f(0,x,v) = f_0(x,v).$$

Assume (3.383) on the scattering kernel $\sigma(g,\theta)$. Let $k > 3/2$ and $\alpha > (3+\beta)/2$. Let $f_0 \in G^0_\alpha(H^k) \cap L^2(L^1)$. Then there exist constants $c_0 > 0$, $c > 0$ such that whenever

$$\|f_0\|_{G_\alpha(H^k)} + \left\{ \int \left(\int |f_0| \, dx \right)^2 dv \right\}^{1/2} < c_0, \tag{3.385}$$

the Cauchy Problem has a unique global solution

$$f \in C^0\left([0,\infty); G^0_\alpha(H^k)\right) \ \cap \ C^1\left([0,\infty); G^0_{\alpha-1}(H^{k-1})\right)$$

which tends to zero according to the estimate
(3.386)

$$\|f(t)\|_{G_\alpha(H^k)} \le c(1+t)^{-3/4} \left(\|f_0\|_{G_\alpha(H^k)} + \left\{ \int \left(\int |f_0| \, dx \right)^2 dv \right\}^{1/2} \right)$$

for $0 \le t < \infty$.

As in the non-relativistic case the proof begins with the construction of the equations for the 14 moments. We know that the nullspace $N(L)$ of L is spanned by the five functions $\sqrt{\mu}$, $v_j\sqrt{\mu}$ $(j = 1,2,3)$, $\sqrt{1+|v|^2}\,\sqrt{\mu}$. We regard $N(L)$ as a subspace of $L^2(v)$ and we let P_0 be the orthogonal projection onto $N(L)$. The subspace $\tilde{W}$ of 14 moments is defined as the space generated by $N(L)$ and the images of $N(L)$ under the mappings $f(v) \to \hat{v}_j f(v)$ $(j = 1,2,3)$. Thus

$$\tilde{W} = \operatorname{span}\{\sqrt{\mu}\,\varphi_j \mid j = 1,\dots,14\} \tag{3.387}$$

where

$$\begin{aligned} &\varphi_1 = 1,\ \varphi_{j+1} = v_j,\ \varphi_5 = \sqrt{1+|v|^2},\ \varphi_{5+j} = v_j\hat{v}_j, \\ &\varphi_9 = v_1\hat{v}_2,\ \varphi_{10} = v_2\hat{v}_3,\ \varphi_{11} = v_3\hat{v}_1,\ \varphi_{11+j} = \hat{v}_j \quad (j = 1,2,3). \end{aligned} \tag{3.388}$$

Then $\tilde{W} \supset N(L)$ and the operator of multiplication by $\hat{v}\cdot\xi$ maps $N(L)$ into $\tilde{W}$. for each $\xi \in \mathbb{R}^3$. We denote by P the orthogonal projection of $L^2(v)$ onto $\tilde{W}$. An orthonormal basis for $N(L)$ is:

$$\begin{cases} e_1 = \kappa_1\sqrt{\mu}, \quad e_{j+1} = \kappa_{j+1}v_j\sqrt{\mu} \quad (j = 1,2,3), \\ e_5 = \kappa_5(\sqrt{1+|v|^2} - c_5)\sqrt{\mu}, \end{cases} \tag{3.389}$$

where $\kappa_1,\dots,\kappa_5$ are normalization constants and c_5 is chosen so that $(e_5, e_1) = 0$. That is,

$$c_5 = \frac{\int \sqrt{1+|v|^2}\mu\,dv}{\int \mu\,dv}. \tag{3.390}$$

An analogous computation to that earlier in this chapter shows that an orthonormal basis for $\tilde{W}$ is given by $e_1,\dots,e_{14}$ where $e_1,\dots,e_5$ appear in (3.389) and

$$\begin{aligned} e_6 &= \kappa_6[(v_1\hat{v}_1 - 1)\mu^{1/2} - d_6e_5] \\ e_7 &= \kappa_7[(v_2\hat{v}_2 - 1)\mu^{1/2} - d_6e_5 - d_7e_6] \\ e_8 &= \kappa_8[(v_3\hat{v}_3 - 1)\mu^{1/2} - d_6e_5 - d_7e_6 - d_8e_7] \\ e_9 &= \kappa_9v_1\hat{v}_2\mu^{1/2},\ e_{10} = \kappa_{10}v_2\hat{v}_3\mu^{1/2},\ e_{11} = \kappa_{11}v_3\hat{v}_1\mu^{1/2}, \\ e_{11+j} &= \kappa_{11+j}(\hat{v}_j - c_{11+j}v_j)\mu^{1/2} \qquad (j = 1,2,3) \end{aligned}$$

where the κ_j are normalization constants and $d_6, d_7, d_8, c_{12}, c_{13}, c_{14}$ are other constants.

Again in analogy to the classical situation, we now project the Boltzmann equation onto $\tilde{W}$ and write the result as an equation for the $W_k = (f, e_k)$. Indeed, let $f(t,x,v)$ satisfy the linear equation

$$\partial_t f + \hat{v}\cdot\nabla_x f + Lf = g, \quad Lf = \nu(v)f + Kf. \tag{3.391}$$

Then, formally, $W = [W_1, \dots, W_{14}]^T$ satisfies the equation

$$\partial_t W + \sum_{i=1}^{3} V^i \partial_{x_i} W + \bar{L} W = \bar{g} + R \tag{3.392}$$

where V^i $(i = 1, 2, 3)$ and $\bar{L}$ are the symmetric matrices

$$V^i = \left\{ (\hat{v}_i\, e_j,\, e_k) \right\}_{j,k=1}^{14}, \quad \bar{L} = \left\{ (L\, e_j,\, e_k) \right\}_{j,k=1}^{14}, \tag{3.393}$$

$\bar{g}$ is the vector with components (g, e_k), and R is a sum of terms involving $(I - P)f$.

Because our goal is to take the Fourier transform of equation (3.391) with respect to x, we study the 14×14 symmetric matrix with entries

$$(V(\xi))_{jk} = ((\hat{v} \cdot \xi) e_j, e_k) \qquad (1 \leq j, k \leq 14). \tag{3.394}$$

Explicitly, the matrix $V(\xi) = \sum_{i=1}^{3} V^i \xi_i$ has the following structure. We write it as

$$V(\xi) = \begin{bmatrix} V_{11}(\xi) & V_{12}(\xi) \\ V_{21}(\xi) & V_{22}(\xi) \end{bmatrix} \tag{3.395}$$

where V_{11} is 5×5, V_{12} is 5×9, V_{21} is 9×5 and V_{22} is 9×9, the submatrices V_{11} and V_{22} are symmetric and $V_{21}^T = V_{12}$. Furthermore,

$$V_{11}(\xi) = \begin{bmatrix} 0 & m\xi_1 & m\xi_2 & m\xi_3 & 0 \\ m\xi_1 & 0 & 0 & 0 & a\xi_1 \\ m\xi_2 & 0 & 0 & 0 & a\xi_2 \\ m\xi_3 & 0 & 0 & 0 & a\xi_3 \\ 0 & a\xi_1 & a\xi_2 & a\xi_3 & 0 \end{bmatrix}$$

$$V_{21}(\xi) = \begin{bmatrix} 0 & a_{11}\xi_1 & a_{12}\xi_2 & a_{13}\xi_3 & 0 \\ 0 & 0 & a_{22}\xi_2 & a_{23}\xi_3 & 0 \\ 0 & 0 & 0 & a_{33}\xi_3 & 0 \\ 0 & b\xi_2 & b\xi_1 & 0 & 0 \\ 0 & 0 & b\xi_3 & b\xi_2 & 0 \\ 0 & b\xi_3 & 0 & b\xi_1 & 0 \\ d'\xi_1 & 0 & 0 & 0 & d\xi_1 \\ d'\xi_2 & 0 & 0 & 0 & d\xi_2 \\ d'\xi_3 & 0 & 0 & 0 & d\xi_3 \end{bmatrix}$$

where m, a, b, d', a_{11}, a_{22} and a_{33} are positive constants and $d < 0$.

At this point in the argument the compensating function itself can be constructed. Its use is the same as in the non–relativistic case, so we omit it.

We conclude this section with two arguments special to the relativistic case. We show that the Maxwellian parameters can indeed be determined in the case of periodic boundary conditions. Then we show that the scattering angle θ from (3.357) is well-defined.

For the determination of the Maxwellian parameters, we must solve this problem: we are given a smooth nonnegative function $F^0(x, v)$, periodic in x and decaying in v. We seek 5 parameters $a \in \mathbb{R}$, $b \in \mathbb{R}^3$ and $c > |b|$ such that the function

$$\mu(v) \equiv \exp(a + b \cdot v - c\sqrt{1 + |v|^2})$$

satisfies (with $B = (0, 2\pi)^3$)

$$\begin{aligned} 0 &= \int_B \int_{\mathbb{R}^3} (F^0 - \mu)\, dx\, dv \\ &= \int_B \int_{\mathbb{R}^3} v(F^0 - \mu)\, dx\, dv \\ &= \int_B \int_{\mathbb{R}^3} \sqrt{1 + |v|^2}\, (F^0 - \mu)\, dx\, dv. \end{aligned}$$

We write $e^a = \alpha$. Then we need to solve

(3.396)
$$\begin{aligned} A &= \int \int F^0\, dx\, dv = (2\pi)^3 \alpha \int e^{b \cdot v - c\sqrt{1+|v|^2}}\, dv, \\ B &= \int \int vF^0\, dx\, dv = (2\pi)^3 \alpha \int v e^{b \cdot v - c\sqrt{1+|v|^2}}\, dv, \\ C &= \int \int \sqrt{1 + |v|^2}\, F^0\, dx\, dv = (2\pi)^3 \alpha \int \sqrt{1 + |v|^2}\, e^{b \cdot v - c\sqrt{1+|v|^2}}\, dv \end{aligned}$$

for $\alpha = e^a, b, c$.

THEOREM 3.15.3 ([29]) *The nonlinear system (3.396) always has a solution a. b. c.*

Proof. First we show that the relation

$$A^2 + |B|^2 < C^2$$

among the given parameters must hold. For this purpose, consider the measure

$$d\mu = \mu_0 \exp(b \cdot v - c\sqrt{1 + |v|^2})\, dv$$

where $b \in \mathbb{R}^3, c > |b|$ and μ_0 is a constant for which $\int_{\mathbb{R}^3} d\mu = 1$. Then $A^2 + |B|^2 < C^2$ is the same as the inequality

$$\left(\int d\mu\right)^2 + \left|\int v\, d\mu\right|^2 < \left(\int \sqrt{1 + |v|^2}\, d\mu\right)^2,$$

i.e.,

$$1 + \left|\int v\, d\mu\right|^2 < \left(\int \varphi(|v|)\, d\mu\right)^2$$

where $\varphi(s) = \sqrt{1+s^2}$ is strictly convex. Since $|\int v\,d\mu| \le \int |v|\,d\mu$, it suffices to show that

$$\sqrt{1+\left(\int |v|\,d\mu\right)^2} < \int \varphi(|v|)\,d\mu.$$

This is just Jensen's inequality, and we are done. Now we show that the condition $A^2+|B|^2<C^2$ is sufficient for existence.

Define $I(b,c) = (2\pi)^3 \int e^{b\cdot v - c\sqrt{1+|v|^2}}dv$. Then I depends only on $|b|$ and c; we write $I(b,c) = \tilde{I}(|b|,c)$ and calculate

$$\begin{aligned}\tilde{I}(|b|,c) &= (2\pi)^3 \int_0^\infty \int_0^\pi \rho^2 \sin\theta e^{|b|\rho\cos\theta - c\sqrt{1+\rho^2}}\,d\theta\,d\rho \\ &= \frac{2(2\pi)^3}{|b|}\int_0^\infty \rho\sinh(\rho|b|)e^{-c\sqrt{1+\rho^2}}d\rho.\end{aligned}$$

The 3 equations can be rewritten as

$$A = \alpha\tilde{I}(|b|,c) \tag{3.397}$$

$$B = \alpha\nabla_b I = \alpha\frac{b}{|b|}\frac{\partial\tilde{I}}{\partial|b|} \tag{3.398}$$

$$C = -\alpha\frac{\partial I}{\partial c} = -\alpha\frac{\partial\tilde{I}}{\partial c}. \tag{3.399}$$

We choose b parallel to B. Then (3.398) becomes a scalar equation, and we are reduced to 3 equations in 3 unknowns α, $|b|$ and c. Then by (3.398) we have $b\cdot B = \alpha|b|\frac{\partial\tilde{I}}{\partial|b|}$, or

$$\pm|B| = \alpha\frac{\partial\tilde{I}}{\partial|b|}. \tag{3.400}$$

Now we can evaluate $\tilde{I}$ in terms of Bessel functions. For fixed $|b|$ and $c>|b|$, consider

$$\begin{aligned}\frac{|b|}{16\pi^3}\int_c^\infty \tilde{I}(b,r)\,dr &= \int_0^\infty \rho\sinh(\rho|b|)e^{-c\sqrt{1+\rho^2}}\frac{d\rho}{\sqrt{1+\rho^2}} \\ &= \int_0^\infty \rho^{\frac{3}{2}}|b|^{\frac{1}{2}}\frac{\sinh(\rho|b|)}{\rho^{1/2}|b|^{1/2}}\cdot\frac{e^{-c\sqrt{1+\rho^2}}}{c^{1/2}(1+\rho^2)^{1/4}}\cdot c^{1/2}(1+\rho^2)^{1/4}\frac{d\rho}{\sqrt{1+\rho^2}} \\ &= \sqrt{|b|c}\int_0^\infty \frac{\rho^{3/2}}{(1+\rho^2)^{1/4}}I_{\frac{1}{2}}(\rho|b|)K_{\frac{1}{2}}(c\sqrt{1+\rho^2})\,d\rho \\ &= e^{-i\frac{\pi}{4}}\sqrt{|b|c}\int_0^\infty \frac{\rho^{\frac{3}{2}}}{(1+\rho^2)^{1/4}}J_{\frac{1}{2}}(i\rho|b|)K_{\frac{1}{2}}(c\sqrt{1+\rho^2})\,d\rho\end{aligned}$$

If the argument of the function $J_{\frac{1}{2}}(\dots)$ here were real, the integral would be known (cf. [33], p. 706, #7). Under the assumption that $c>|b|$ (to be verified below at the solution) we can justify an analytic continuation to complex arguments, and get for the right-hand side

$$
\begin{aligned}
&e^{-i\frac{\pi}{4}}\sqrt{|b|c}\cdot e^{i\frac{\pi}{4}}\sqrt{\frac{|b|}{c}}\cdot\frac{1}{\sqrt{c^2-|b|^2}}K_{-1}(\sqrt{c^2-b^2})\\
&=\frac{|b|}{\sqrt{c^2-|b|^2}}K_1(\sqrt{c^2-|b|^2})
\end{aligned}
$$

since $K_{-\nu}=K_\nu$. Now call $x=\sqrt{c^2-|b|^2}$. Differentiating with respect to c, we get

$$
\begin{aligned}
-\frac{|b|}{16\pi^3}\tilde{I}(|b|,c) &= |b|\frac{\partial}{\partial c}\left[\frac{K_1(x)}{x}\right]\\
&= |b|\frac{d}{dx}[x^{-1}K_1(x)]\frac{c}{x}\\
&= -\frac{|b|c}{x}\cdot x^{-1}K_2(x)
\end{aligned}
$$

and hence

$$\tilde{I}(|b|,c)=16\pi^3\frac{cK_2(x)}{x^2}. \tag{3.401}$$

We abbreviate this by

$$\tilde{I}(|b|,c)=cg(\sqrt{c^2-|b|^2})$$

with $g(y)=16\pi^3y^{-2}K_2(y)$. Thus

$$
\begin{aligned}
\frac{\partial\tilde{I}}{\partial|b|} &= cg'(x)\frac{dx}{d|b|} = -\frac{|b|c}{x}g'(x);\\
\frac{\partial\tilde{I}}{\partial c} &= g(x)+cg'(x)\frac{dx}{dc} = g(x)+\frac{c^2}{x}g'(x).
\end{aligned}
$$

We know that $\frac{d}{dx}(x^{-2}K_2(x)) = -x^{-2}K_3(x)\le 0$. Hence $\frac{\partial\tilde{I}}{\partial|b|}\ge 0$ so that equations (3.397)–(3.399) become

$$A=\alpha\ c\ g(x) \tag{3.402}$$

$$|B|=-\frac{\alpha|b|c}{x}g'(x) \tag{3.403}$$

$$C=-\alpha\left(g(x)+\frac{c^2}{x}g'(x)\right). \tag{3.404}$$

Dividing the last two equations by the first, we get

$$\frac{|B|}{A}=-\frac{|b|}{x}\cdot\frac{g'(x)}{g(x)}; \tag{3.405}$$

$$-\frac{C}{A}=\frac{1}{c}+\frac{c}{x}\frac{g'(x)}{g(x)}. \tag{3.406}$$

Now solve (3.405) for $\frac{g'}{g}$ and put the result into (3.406):

$$-\frac{C}{A} = \frac{1}{c} + \frac{c}{x}\left[-\frac{x|B|}{|b|A}\right],$$

so that

$$|b| = |b(c)| = \frac{c^2|B|}{A + Cc}. \tag{3.407}$$

Since $A > 0$ and $C \geq |B|$,

$$|b(c)| = \frac{c^2|B|}{A + Cc} < \frac{c^2|B|}{Cc} \leq c.$$

Thus any solution c generates a solution $|b| = |b(c)|$ for which the condition $|b| < c$ holds.

It remains to solve for c. By recursions for the Bessel functions.

$$\frac{g'(x)}{g(x)} = \frac{(x^{-2}K_2(x))'}{x^{-2}K_2(x)} = -\frac{x^{-2}K_3(x)}{x^{-2}K_2(x)} = -\frac{K_3(x)}{K_2(x)}.$$

Therefore (3.406) can be rewritten as

$$\frac{xK_2(x)}{K_3(x)} = \frac{Ac^2}{A + Cc}. \tag{3.408}$$

Since $|b| = |b(c)|$ is known from (3.407), (3.408) is an equation in c alone. In fact, by definition and (3.407),

$$x^2 \equiv c^2 - |b|^2 = c^2 - \frac{c^4|B|^2}{(A + cC)^2}$$

so that

$$x = x(c) = \frac{c}{A + Cc}\sqrt{(A + Cc)^2 - |B|^2c^2}\ . \tag{3.409}$$

Now consider (3.408) rewritten as $q(c) = 0$, where

$$q(c) = \frac{Ac^2}{A + Cc} - \frac{x(c)K_2(x(c))}{K_3(x(c))}. \tag{3.410}$$

From (3.409) we see that

$$\lim_{c\to 0^+} \frac{x(c)}{c} = 1, \quad \lim_{c\to\infty} \frac{x(c)}{c} = \frac{\sqrt{C^2 - |B|^2}}{C}\ .$$

From the series definitions of the Bessel functions, we have near $y = 0$,

$$\frac{yK_2(y)}{K_3(y)} \sim \frac{y \cdot \frac{1}{2}\left(\frac{y}{2}\right)^{-2}}{\left(\frac{y}{2}\right)^{-3}} = \left(\frac{y}{2}\right)^2.$$

Thus as $c \to 0$,

$$q(c) \sim c^2 - \frac{1}{4}x(c)^2 \sim \frac{3}{4}c^2$$

and therefore $q(c) > 0$ for small $c > 0$.

For large arguments, both $K_2(y)$ and $K_3(y)$ have the asymptotic representations $\sqrt{\frac{\pi}{2y}}\, e^{-y}$. As $c \to \infty$ then,

$$q(c) \sim \frac{A}{C}c \; - x(c) \sim c\left[\frac{A}{C} - \frac{\sqrt{C^2 - |B|^2}}{C}\right].$$

As was shown at the beginning of the proof, this is negative. Since q is continuous in $\{c > 0\}$, we are done.

Lastly, we show that the scattering angle defined by

$$\cos\theta = \frac{(V-U)\cdot(V'-U')}{(V-U)^2}. \tag{3.411}$$

is well-defined.

We are dealing with 4-vectors

$$V = (v_0, v_1, v_2, v_3) \equiv (v_0, v)$$

where $v_0 \geq 1$ and the Lorentz inner product is given via

$$V \cdot V \equiv v_0^2 - |v|^2 = 1. \tag{3.412}$$

The scattered variables U', V' satisfy (3.412) and

$$U + V = U' + V'. \tag{3.413}$$

LEMMA 3.15.1 *Let $V \cdot V = U \cdot U = 1$. Then*

i) $U \cdot V > 1$
ii) $(U - V)\cdot(U - V) < 0$

Proof. We have

$$1 + |u|^2 + |v|^2 + |u|^2|v|^2 > 1 + 2u\cdot v + (u\cdot v)^2$$

and hence

$$u_0^2 v_0^2 = (1 + |u|^2)(1 + |v|^2) > (1 + u \cdot v)^2$$

so that

$$u_0 v_0 > |1 + u\cdot v| \geq 1 + u \cdot v.$$

Equality holds only if u and v are linearly dependent. However, this cannot occur since both lie on the hyperboloid.

For the second assertion we have

$$(U - V) \cdot (U - V) = 2(1 - U \cdot V) < 0$$

by the first part. This assertion means that $U - V$ is spacelike.

LEMMA 3.15.2 *Let V, U, V' and U' satisfy (3.412) and (3.413). Then*

i) $(U - V) \cdot (U - V) = (U' - V') \cdot (U' - V')$
ii) $U \cdot V = U' \cdot V'$
iii) $U \cdot U' = V \cdot V'$

Proof. The parallelogram law gives us

$$\begin{aligned}(U - V) \cdot (U - V) &= 2U \cdot U + 2V \cdot V - (U + V) \cdot (U + V) \\ &= 2U' \cdot U' + 2V' \cdot V' - (U' + V') \cdot (U' + V') \\ &= (U' - V') \cdot (U' - V').\end{aligned}$$

This is the first assertion. For the second, we have

$$\begin{aligned}(U - V) \cdot (U - V) &= U \cdot U + V \cdot V - 2U \cdot V \\ &= 2 - 2U \cdot V\end{aligned}$$

and similarly

$$(U' - V') \cdot (U' - V') = 2 - 2U' \cdot V'.$$

Now the second assertion follows from the first.

Lastly, by (3.413) we get

$$(U - U') \cdot (U - U') = (V' - V) \cdot (V' - V).$$

Expanding this, we have

$$1 + 1 - 2U \cdot U' = 1 + 1 - 2V' \cdot V$$

as desired.

Using (3.413) we can write

$$\begin{aligned}(V - U) \cdot (V' - U') &= -(V + U) \cdot (V' + U') + 2V \cdot V' + 2U \cdot U' \\ &= -(V + U) \cdot (V + U) + 2V \cdot V' + 2U \cdot U' \\ &= (V - U)^2 - 2V \cdot V - 2U \cdot U + 2V \cdot V' + 2U \cdot U'\end{aligned}$$

where we have used the parallelogram law in the last step. Therefore

$$(V - U) \cdot (V' - U') = (V - U) \cdot (V - U) - 4 + 4V \cdot V'. \tag{3.414}$$

Now we are in a position to estimate the scattering angle. Recall that

$$s - 4 = -(V - U) \cdot (V - U) \equiv 4g^2 > 0. \tag{3.415}$$

Lemma 3.15.3: *Let the scattering angle be defined as in (3.411) above. Then we have*

$$|\cos\theta| < 1.$$

Proof. Applying part (1) of Lemma 3.15.1 to the vectors V and V', we have $V \cdot V' > 1$. Hence by (3.414)

$$(V - U) \cdot (V' - U') > (V - U) \cdot (V - U).$$

By part (2) of Lemma 3.15.1 then, $\cos\theta < 1$.

Now we claim that $\cos\theta > -1$. By (3.411) and (3.414) we have

$$\cos\theta = \frac{(V - U) \cdot (V' - U')}{(V - U) \cdot (V - U)} = 1 + \frac{4(V \cdot V' - 1)}{(V - U) \cdot (V - U)}.$$

Thus $\cos\theta > -1$ if and only if

$$\frac{4(V \cdot V' - 1)}{(V - U) \cdot (V - U)} > -2. \tag{3.416}$$

(3.416) holds whenever

$$V \cdot V' - 1 < -\frac{1}{2}(V - U) \cdot (V - U) = U \cdot V - 1$$

and this is valid whenever

$$V \cdot V' < V \cdot U, \quad \text{i.e.,} \quad V \cdot (V' - U) < 0. \tag{3.417}$$

By the conservation law $U + V = U' + V'$, this is the same as

$$V \cdot (V - U') < 0, \quad \text{or} \quad 1 - V \cdot U' < 0.$$

This relation is valid by part (1) of Lemma 3.15.1.

This exposition is due in large part to W. Strauss.

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Chapter 4

THE VLASOV–POISSON SYSTEM

4.1. Introduction

The *Vlasov–Poisson* system (VP) can be written as

$$
\begin{aligned}
(4.1) \qquad & f_t + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0 \quad (x, v \in \mathbb{R}^3); \\
& \rho(t,x) = \int f \, dv, \\
& E(t,x) = \frac{1}{4\pi} \int \frac{(x-y)\rho(t,y)\, dy}{|x-y|^3}, \\
& f(0,x,v) = f_0(x,v).
\end{aligned}
$$

Note that $E = \nabla_x u$, $u = -\frac{1}{4\pi r} * \rho$, $\Delta u = \rho \quad (r \equiv |x|)$. We take

$$\gamma = \begin{cases} +1 & \text{for plasma problems} \\ -1 & \text{for astrophysics problems.} \end{cases}$$

(In plasma physics problems, we should actually have several species (as written in Chapter 1), or at least a neutralizing background density.)

Here is a brief history of the mathematical solution of the Cauchy Problem for (VP). The first paper on global existence is due to Arsen'ev [1]. He showed global existence of weak solutions. Then in 1977 Batt [5] established global existence for spherically symmetric data. In 1981 Horst [27] extended global classical solvability to cylindrically symmetric data. Next, in 1985, Bardos and Degond [2] obtained global existence for "small" data. Finally, in 1989 Pfaffelmoser [34] proved the global existence of a smooth solution with large (unrestricted size) data. Later, in 1991, simpler proofs of the same were published by Schaeffer [41], Horst [28], and Lions and Perthame [31].

A basic observation is the following. Take f_0 smooth and of compact support in (x,v), and assume local existence. Put

$$Q(t) = 1 + \sup\{|v| : \text{there exists } x \in \mathbb{R}^3,\ \tau \in [0,t] \text{ such that } \ f(\tau,x,v) \neq 0\}.$$

A by-product of local existence is this: f can be continued to $[0,T]$ (T arbitrary) provided

$$(4.2) \qquad Q(t) \le c_T \quad \text{for} \quad 0 \le t \le T.$$

See Batt [5] or Horst [27]. Thus we need to control the influence of large velocities.

Here is a statement of Schaeffer's Theorem [41].

THEOREM 4.1.1 *Let* $0 \le f_0 \in C_0^1$, $\gamma = \pm 1$. *Then the Cauchy Problem for* (VP) *has a unique* C^1*-solution, and for any* $p > \frac{33}{17}$, *there exists a constant* c_p *such that*

$$Q(t) \le c_p(1+t)^p.$$

The rate of growth of Q in time has been improved in [28] to $Q(t) \le ct\ln^{11/14} t$ for large t.

4.2. Preliminaries and A Priori Estimates

4.2.1. The Characteristics. The characteristics are the solutions to

$$\begin{aligned} \frac{dX}{ds} &= V, \quad X(t,t,x,v) = x \\ \frac{dV}{ds} &= \gamma E, \quad V(t,t,x,v) = v. \end{aligned} \tag{4.3}$$

Since

$$\frac{d}{ds} f(s, X(s,t,x,v), V(s,t,x,v)) = f_t + \nabla_x f \cdot \dot{X} + \nabla_v f \cdot \dot{V} = 0$$

we have

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v)) \ge 0 \tag{4.4}$$

and $\sup_{x,v} f \le \|f_0\|_\infty$, assuming that f_0 is nonnegative and bounded.

4.2.2. The Measure Preserving Property. The map

$$(x,v) \mapsto (X(s,t,x,v), V(s,t,x,v))$$

is a measure preserving homeomorphism. Assume this for now. For $\sigma \in L^1_{\text{loc}}$, consider

$$I \equiv \iint f(s,y,w)\sigma(y,w)\, dw\, dy.$$

Change variables by

$$\begin{aligned} y &= X(s,t,x,v) \\ w &= V(s,t,x,v). \end{aligned}$$

Then the Jacobian equals 1 and

$$\begin{aligned} x &= X(t,s,y,w) \\ v &= V(t,s,y,w). \end{aligned}$$

Thus

$$(4.5)\, I = \iint f(s, X(s,t,x,v), V(s,t,x,v))\sigma(X(s,t,x,v), V(s,t,x,v))\, dv\, dx$$
$$= \iint f(t,x,v)\sigma(X(s,t,x,v), V(s,t,x,v))\, dv\, dx.$$

Hence

(4.6)
$$\iint f(s,y,w)\sigma(y,w)\, dw\, dy = \iint f(t,x,v)\sigma(X(s,t,x,v), V(s,t,x,v))\, dv\, dx.$$

Similarly, let

$$J \equiv \iint f(t,x,v)\sigma(x,v)\, dv\, dx$$

and set

$$x = X(t,s,y,w)$$
$$v = V(t,s,y,w)$$

Again this map has unit Jacobian and

$$y = X(s,t,x,v)$$
$$w = V(s,t,x,v).$$

Thus

$$\begin{aligned} J &= \iint f(t, X(t,s,y,w), V(t,s,y,w)) \cdot \\ &\qquad \sigma(X(t,s,y,w), V(t,s,y,w))\, dw\, dy \\ &= \iint f(s,y,w)\sigma(X(t,s,y,w), V(t,s,y,w))\, dw\, dy. \end{aligned} \tag{4.7}$$

Now we return to the Measure Preserving Property. Actually this follows directly because (4.3) is a Hamiltonian system, but we will proceed via a computation. We need to evaluate the Jacobian $\left|\frac{\partial(X,V)}{\partial(x,v)}\right|$. We compute the variational Ordinary Differential Equations: put

$$X \to X + \epsilon y \qquad V \to V + \epsilon w :$$

$$\begin{aligned} &\frac{d}{ds}(X + \epsilon y) = V + \epsilon w \\ &\frac{d}{ds}(V + \epsilon w) = \gamma E(s, X + \epsilon y) \end{aligned} \tag{4.8}$$

Take $\frac{\partial}{\partial \epsilon}$ and set $\epsilon = 0$ to get

$$\begin{aligned} &\dot{y} = w \\ &\dot{w} = \gamma E_X(s,X)y. \end{aligned} \tag{4.9}$$

We abbreviate this as $\dot{u} = A(s)u$ where $u = [y\ w]^T$.

Let $U(t)$ be the fundamental matrix solution of (4.9). Recall that

$$\begin{aligned}\det U(t) &= \det U(0) \exp \int_0^t tr\, A(s)\, ds \\ &= \det U(0) \cdot e^0 = \det U(0) = 1\end{aligned}$$

because the trace of A vanishes identically. Hence this mapping preserves measure.

4.2.3. The Energy. We have

$$|v|^2 f_t + |v|^2 v \cdot \nabla_x f + \gamma |v|^2 \nabla_v \cdot (Ef) = 0 \tag{4.10}$$

and hence, assuming that f has compact support or vanishes with sufficient rapidity at infinity, we find by formal integration

$$\partial_t \iint |v|^2 f\, dv\, dx = 2\gamma \iint v \cdot Ef\, dv\, dx = 2\gamma \int j \cdot E\, dx \tag{4.11}$$

where

$$j(t,x) = \int v f(t,x,v)\, dv. \tag{4.12}$$

Now

$$\rho_t + \nabla \cdot j = 0 \tag{4.13}$$

follows from integrating (VP) over v. Hence $\|\rho(t)\|_1 = \text{const}$. Further, we compute

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int |E|^2\, dx &= \int E \cdot E_t\, dx = \int \nabla u \cdot \nabla u_t\, dx \\ &= -\int u \Delta u_t\, dx = -\int u \rho_t\, dx = \int u \nabla \cdot j\, dx \\ &= -\int j \cdot \nabla u\, dx = -\int j \cdot E\, dx.\end{aligned} \tag{4.14}$$

Thus

$$\iint |v|^2 f\, dv\, dx + \gamma \int |E|^2\, dx = \text{const}. \tag{4.15}$$

For the plasma sign, $\gamma = +1$, we get automatic bounds on both quantities. Now we make the **claim** that even if the energy is indefinite (i.e., if $\gamma = -1$) we still have

$$\iint |v|^2 f\, dv\, dx \le \text{const}. \tag{4.16}$$

This observation is due to Horst [25] who also showed that in this case ($\gamma = -1$) the solution can blow up in finite time on $\mathbb{R}^n$, $n \geq 4$. See the end of this chapter. In order to establish the claim, note that

$$|E(t,x)| \leq cr^{-2} * \rho(t,x) \quad (r = |x|).$$

Recall that

$$\|r^{-\lambda} * \phi\|_p \leq c\|\phi\|_q \quad \text{on} \quad \mathbb{R}^n$$

where

$$0 < \frac{1}{p} = \frac{1}{q} + \frac{\lambda}{n} - 1.$$

We take $\lambda = 2$, $n = 3$, $p = 2$. Then $q = 6/5$ and hence

$$\|E(t)\|_2 \leq c\|r^{-2} * \rho(t,\cdot)\|_2 \leq c\|\rho(t)\|_{6/5} \leq c\|\rho(t)\|_1^{\frac{7}{12}} \|\rho(t)\|_{5/3}^{\frac{5}{12}}. \tag{4.17}$$

Therefore $\|E(t)\|_2 \leq c\|\rho(t)\|_{5/3}^{5/12}$ because $\|\rho(t)\|_1 = \text{const}$. Now write

$$\begin{aligned}\rho = \int f\,dv &\leq \int_{|v|<R} f\,dv + R^{-2}\int_{|v|>R} |v|^2 f\,dv \\ &\leq \frac{4\pi}{3}\|f_0\|_\infty R^3 + R^{-2}\,\overline{e}(t,x)\end{aligned}$$

where

$$\overline{e}(t,x) = \int |v|^2 f\,dv.$$

We choose R via $R^3 = R^{-2}\cdot\overline{e}$, or $R = \overline{e}^{1/5}$. Then

$$\rho \leq cR^3 = c\left(\int |v|^2 f\,dv\right)^{3/5} \tag{4.18}$$

Thus

$$\int \rho^{5/3}\,dx \leq c\iint |v|^2 f\,dv\,dx \tag{4.19}$$

and hence

$$\|E(t)\|_2 \leq c\left(\iint |v|^2 f\,dv\,dx\right)^{\frac{3}{5}\cdot\frac{5}{12}} = \left(\iint |v|^2 f\,dv\,dx\right)^{1/4} \tag{4.20}$$

so that

$$\int |E|^2\,dx \leq c\left(\iint |v|^2 f\,dv\,dx\right)^{1/2}. \tag{4.21}$$

Therefore even if $\gamma = -1$, $\iint |v|^2 f\,dv\,dx \leq \text{const}$.

4.2.4. Bounds on the Field. A standard estimate on the gradient of a Newtonian potential (with $\Delta u = \rho$, $E = \nabla_x u$) is

$$\|E(t)\|_\infty \le c\|\rho(t)\|_\infty^{4/9}\|\rho(t)\|_{5/3}^{5/9}. \tag{4.22}$$

For the proof, write

$$\begin{aligned}
4\pi|E(t,x)| &\le \int \frac{\rho(t,y)\,dy}{|x-y|^2} \\
&\le \|\rho(t)\|_\infty \int_{|y-x|<R} \frac{dy}{|y-x|^2} + \|\rho(t)\|_{5/3}\left(\int_{|y-x|>R}\frac{dy}{|x-y|^{2\cdot\frac{5}{2}}}\right)^{2/5} \\
&\le c\|\rho(t)\|_\infty \cdot R + c\|\rho(t)\|_{5/3}\left(\int_R^\infty r^{2-5}dr\right)^{2/5} \\
&= c\|\rho(t)\|_\infty R + c\|\rho(t)\|_{5/3}R^{-4/5}.
\end{aligned}$$

Set $\|\rho(t)\|_\infty R = \|\rho(t)\|_{5/3}R^{-4/5}$. Then

$$|E(t,x)| \le cR\cdot\|\rho(t)\|_\infty \le c\|\rho(t)\|_\infty^{4/9}\cdot\|\rho(t)\|_{5/3}^{5/9}$$

as desired.

4.2.5. Estimates on the Derivatives of the Field. Let $0 \le \rho \in L^1$ be Lipschitz. Let Lip ρ be the Lipschitz constant for ρ. Then for $0 < d \le R$,

$$\left|\frac{\partial E}{\partial x_k}(t,x)\right| \le c\,(1+\ln(R/d)) \sup_{|y-x|\le R}\rho(t,y) + cd\,\mathrm{Lip}\,\rho(t,\cdot) + cR^{-3}\|\rho(t)\|_1. \tag{4.23}$$

Proof. ([5]) Classical differentiability of potentials implies

$$\begin{aligned}
-\frac{\partial E^k}{\partial x_k}(t,x) &= -\frac{1}{3}\rho(t,x) + \frac{1}{4\pi}\int_{|y-x|\le d}(\rho(t,y)-\rho(t,x))\left(\frac{3(y_k-x_k)^2}{r^5}-\frac{1}{r^3}\right)dy \\
&\quad + \left(\frac{1}{4\pi}\int_{d\le|y-x|\le R}\rho(t,y) + \frac{1}{4\pi}\int_{|y-x|\ge R}\rho(t,y)\right)\left(\frac{3(y_k-x_k)^2}{r^5}-\frac{1}{r^3}\right)dy
\end{aligned}$$

where $r = |y-x|$. For the second term (involving the integral of $\rho(t,y)-\rho(t,x)$) we have the bound

$$\text{const. }(\mathrm{Lip}\,\rho(t,\cdot))\cdot\int_{|y-x|\le d}\frac{dy}{r^2} \le cd\cdot\mathrm{Lip}\,\rho(t,\cdot).$$

The third term (involving the integral of $\rho(t,y)$ over $d \le |y-x| \le R$) is less than

$$\text{const. }\|\rho(t)\|_\infty\int_{d\le|y-x|\le R}\frac{dy}{r^3} = c\|\rho(t)\|_\infty\int_d^R\frac{dr}{r} = c\ln\frac{R}{d}\cdot\|\rho(t)\|_\infty$$

and the fourth term (involving the integral of $\rho(t,y)$ over $|y-x|>R$) is dominated by

$$\text{const.}\, R^{-3}\|\rho(t)\|_1.$$

Similarly, for $i \neq k$,

$$(4.24)\quad -\frac{\partial E^k}{\partial x_i}(t,x) = \int_{|y-x|\le d} (\rho(t,y)-\rho(t,x))\,\frac{3(y_i-x_i)(y_k-x_k)\,dy}{r^5} + \int_{|y-x|\ge d} \rho(t,y)\frac{3(y_i-x_i)(y_k-x_k)}{r^5}\,dy$$

which satisfies the same estimate.

COROLLARY: *Suppose also that $\|\rho(t)\|_\infty \le c_T < \infty$ on $t \le T$. Define*

$$\ln^* s = \begin{cases} s & 0 \le s \le 1 \\ 1+\ln s & s > 1. \end{cases}$$

Then for a solution to (VP)

$$(4.25)\qquad \sup_x |D_x E(t,x)| \le c_T\left(1+\ln^*(\sup_x |D_x\rho|)\right).$$

Proof. Simply replace Lip_ρ by $\sup_x |D_x\rho|$, and take $d = \frac{1}{\sup_x |D_x\rho|}$.

Suppose that the v–support of f is bounded for bounded times. Then there exists $Q(t)$ such that

$$(4.26)\qquad \rho = \int f\,dv = \int_{|v|\le Q(t)} f\,dv \le cQ(t)^3$$

and

$$|D_x\rho| \le cQ(t)^3 \sup_{x,v} |f_x(t,x,v)|.$$

Therefore from (4.16), (4.19) and (4.22) we have

$$(4.27)\qquad \|E\|_\infty \le c\|\rho\|_\infty^{4/9} \le c_1 Q(t)^{4/3}$$

and $|E_x|$ grows at most like $\ln(c_T \sup_{x,v}|f_x|)$ for $|f_x|$ large $\quad(t \le T)$.

4.2.6. Estimates on the Derivatives of the Density. For $0 < t < T$, T arbitrary, let f satisfy

$$(4.28)\qquad \partial_t f + v\cdot\nabla_x f + \gamma E\cdot\nabla_v f = 0; \quad f|_{t=0} = f_0.$$

Let D be any x derivative. Then

$$(4.29)\qquad \partial_t(Df) + v\cdot\nabla_x(Df) + \gamma E\cdot\nabla_v(Df) = -\gamma DE\cdot\nabla_v f$$

so that

(4.30)
$$\frac{d}{ds}Df(s, X(s,t,x,v), V(s,t,x,v)) = -\gamma DE \cdot \nabla_v f(s, X(s,t,x,v), V(s,t,x,v)).$$

Hence

$$(4.31)\ |Df(t,x,v)| \le |Df(0, X(0,t,x,v), V(0,t,x,v))| + \int_0^t |DE \cdot \nabla_v f(s, X(s,t,x,v), V(s,t,x,v))|\, ds.$$

Define

$$|f(s)|_1 = \sup_{x,v} |\partial_x f(s,x,v)| + \sup_{x,v} |\partial_v f(s,x,v)|.$$
$$|E(s)|_1 = \sup_{x} |\partial_x E(s,x)|$$

and assume that f_0 has bounded partial derivatives. Then

$$|Df(t,x,v)| \le c_0 + \int_0^t |E(s)|_1 |f(s)|_1\, ds. \tag{4.32}$$

We see that $\partial_v f$ satisfies an inequality of similar type because

$$\partial_t(\partial_v f) + v \cdot \nabla_x(\partial_v f) + \gamma E \cdot \nabla_v(\partial_v f) = -\partial_v v \cdot \nabla_x f.$$

It follows that (with a different constant c_0)

$$|f(t)|_1 \le c_0 + \int_0^t (1 + |E(s)|_1)|f(s)|_1\, ds. \tag{4.33}$$

However, from Section 4.2.5 we know that

$$|E(s)|_1 \le c_T(1 + \ln^* |f(s)|_1) \quad \text{on} \quad s \le T.$$

Therefore for $t \le T$,

$$|f(t)|_1 \le c_T \left[1 + \int_0^t (1 + \ln^* |f(s)|_1)|f(s)|_1\, ds\right]. \tag{4.34}$$

It now follows by an application of the Gronwall inequality that

$$|f(t)|_1 \le c_T \quad \text{on} \quad t \le T,$$

as well as

$$|E(t)|_1 \le c_T \quad \text{on} \quad t \le T.$$

4.3. Sketch of the Existence Proof

Let

$$f_t + v \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0, \quad f(0,x,v) = f_0(x,v).$$

Given $f_0 \geq 0$, compute $\rho_0 = \int f_0 \, dv$; then u_0 via $\Delta u_0 = \rho_0$, then $E_0 = \nabla u_0$. Then we get f_1 by solving $(\partial_t + v \cdot \nabla_x) f_1 + E_0 \cdot \nabla_v f_1 = 0$, $f_1(0,x,v) = f_0(x,v)$, etc.

In general: given f_n, define

$$\rho_n = \int f_n \, dv, \ \ \Delta u_n = \rho_n, \ \ E_n = \nabla u_n. \tag{4.35}$$

Then we define f_{n+1} to be the solution of

$$\frac{\partial}{\partial t} f_{n+1} + v \cdot \nabla_x f_{n+1} + \gamma E_n \cdot \nabla_v f_{n+1} = 0$$

$$f_{n+1}(0,x,v) = f_0(x,v).$$

The characteristics are

$$\dot{X}_{n+1} = V_{n+1}$$
$$\dot{V}_{n+1} = \gamma E_n$$

so the characteristics change with n, but we still have the essential feature that $0 \leq f_n(t,x,v) \leq$ const. because f_n is constant on its characteristics. When one applies the above estimates to these iterates, and addresses the relevant regularity issues, the local existence theorem follows. This iteration is studied in full detail for the more general case of the Vlasov–Maxwell system in the next chapter.

4.4. The Good, the Bad and the Ugly

From now on we can assume that $\gamma = 1$. This is because we have already shown that the kinetic energy is bounded in (4.16) even when $\gamma = -1$, and this is the only ingredient needed in the argument which follows.

Let $(\hat{X}(t), \hat{U}(t))$ be any fixed characteristic:

$$\frac{d}{dt}\hat{X} = \hat{U}, \ \frac{d}{dt}\hat{U} = E(t, \hat{X})$$

for which

$$f(t, \hat{X}(t), \hat{U}(t)) \neq 0.$$

For any $0 \leq \Delta \leq t$, we have

$$\begin{aligned} \int_{t-\Delta}^{t} |E(s, \hat{X}(s))| \, ds &\leq c \int_{t-\Delta}^{t} \iint \frac{f(s,y,w) \, dw \, dy \, ds}{|y - \hat{X}(s)|^2} \\ &= c \int_{t-\Delta}^{t} \iint \frac{f(t,x,v) \, dv \, dx \, ds}{|X(s,t,x,v) - \hat{X}(s)|^2} \end{aligned} \tag{4.36}$$

by (4.6).

Let $0 < P \le Q(t)$, $R > 0$, $\Delta = \frac{P}{4}(c_1 Q^{4/3}(t))^{-1}$, where the bound for the E field is $\|E(t)\|_\infty \le c_1 Q^{4/3}(t)$ from (4.27).

We partition the integral in (4.36) as $\displaystyle\iiint_G + \iiint_B + \iiint_U$ where

$G =$ **good set** $= \{(s,x,v) : t - \Delta < s < t$ and $(|v| < P$ or $|v - \hat{U}(t)| < P)\}$;
$B =$ **bad set** $= \{(s,x,v) : t - \Delta < s < t$ and $|v| > P$ and $|v - \hat{U}(t)| > P$ and $(|X(s,t,x,v) - \hat{X}(s)| < R|v|^{-3}$ or $|X(s,t,x,v) - \hat{X}(s)| < R|v - \hat{U}(t)|^{-3})\}$;
$U =$ **ugly set** $= \{(s,x,v) : t - \Delta < s < t$ and $|v| > P$ and $|v - \hat{U}(t)| > P$ and $|X(s,t,x,v) - \hat{X}(s)| > R|v|^{-3}$ and $|X(s,t,x,v) - \hat{X}(s)| > R|v - \hat{U}(t)|^{-3}\}$.

For the characteristics recall that when we set

$$\begin{aligned} y &= X(s,t,x,v) \\ w &= V(s,t,x,v) \end{aligned}$$

we can invert via

$$\begin{aligned} x &= X(t,s,y,w) \\ v &= V(t,s,y,w). \end{aligned}$$

In particular,

$$w = V(s,t,X(t,s,y,w),V(t,s,y,w)). \tag{4.37}$$

Firstly we make a number of preliminary observations.

1. (A) $|V(s,t,x,v) - v| = |\int_s^t E\,d\tau|$.
   Thus for $s \in [t-\Delta, t]$, $|V(s,t,x,v) - v| \le \Delta \cdot c_1 Q^{4/3}(t) \le P/4$.
   (B) $|\hat{U}(s) - \hat{U}(t)| \le \frac{1}{4}P$ for $s \in [t-\Delta, t]$.
2. (A) for $|v| < P$, $|V(s,t,x,v)| \le |v| + \frac{1}{4}P < 2P$ (by 1(A))
   (B) for $|v| > P$, $\frac{1}{2}|v| < |v| - \frac{|v|}{4} < |v| - \frac{P}{4} \le |V(s,t,x,v)| \le |v| + \frac{P}{4} < 2|v|$.
3. (A) if $|v - \hat{U}(t)| < P$ then
   $|V(s,t,x,v) - \hat{U}(s)|$
   $\le |v - \hat{U}(t)| + |\hat{U}(t) - \hat{U}(s)| + |V(s,t,x,v) - v|$
   $< P + \frac{P}{4} + \frac{P}{4} < 2P$.
   (B) if $|v - \hat{U}(t)| > P$ then

$$\frac{1}{2}|v - \hat{U}(t)| = |v - \hat{U}(t)| - \frac{1}{2}|v - \hat{U}(t)|$$

$$
\begin{aligned}
&< |v - \hat{U}(t)| - \frac{P}{4} - \frac{P}{4} \\
&< |v - \hat{U}(t)| - |V(s,t,x,v) - v| - |\hat{U}(s) - \hat{U}(t)| \\
&\le |V(s,t,x,v) - \hat{U}(s)| \\
&\le |V(s,t,x,v) - v| + |\hat{U}(s) - \hat{U}(t)| + |v - \hat{U}(t)| \\
&\le \frac{P}{4} + \frac{P}{4} + |v - \hat{U}(t)| \\
&\le 2|v - \hat{U}(t)|.
\end{aligned}
$$

We now estimate the integral over the **good set**. Let

$$
I_G = \iiint_G = \iiint_G \frac{f(t,x,v)\,dv\,dx\,ds}{|X(s,t,x,v) - \hat{X}(s)|^2} = \iiint_G \frac{f(s,y,w)\,dw\,dy\,ds}{|y - \hat{X}(s)|^2}.
$$

Let

$$
\chi_G(s,x,v) = \begin{cases} 1 & (s,x,v) \in G \\ 0 & \text{else} \end{cases}
$$

Then

$$
\begin{aligned}
I_G &= \int_{t-\Delta}^{t} \iint \frac{\chi_G(t,x,v) f(t,x,v)\,dv\,dx\,ds}{|X(s,t,x,v) - \hat{X}(s)|^2} \\
&= \int_{t-\Delta}^{t} \iint \frac{f(s,y,w)}{|y - \hat{X}(s)|^2} \cdot \chi_G(s, X(t,s,y,w), V(t,s,y,w))\,dw\,dy\,ds.
\end{aligned}
$$

Now if $\chi_G(\cdots) \neq 0$, then $V(t,s,y,w)$ must satisfy either

$$
|V(t,s,y,w)| < P \quad \text{or} \quad |V(t,s,y,w) - \hat{U}(t)| < P.
$$

Therefore by Preliminary 2(A), 3(A) we have either

$$
|w| < 2P \quad \text{or} \quad |w - \hat{U}(s)| < 2P. \tag{4.38}
$$

Set

$$
\tilde{\rho}(s,y) = \int f(s,y,w)\chi_G(s, X(t,s,y,w), V(t,s,y,w))\,dw.
$$

Then $\|\tilde{\rho}(s)\|_\infty \le cP^3$ by (4.38). Moreover, $0 \le \tilde{\rho} \le \rho$, and hence $\|\tilde{\rho}\|_{5/3} \le \|\rho\|_{5/3} \le c$. Therefore using (4.22) we get

$$
\int \frac{\tilde{\rho}(s,y)}{|y - \hat{X}(s)|^2}\,dy \le c\|\tilde{\rho}(s)\|_\infty^{4/9} \|\tilde{\rho}(s)\|_{5/3}^{5/9} \le cP^{4/3}.
$$

Thus we have

$$
I_G \le c\Delta \cdot P^{4/3}. \tag{4.39}
$$

Now we turn to the estimation of the integral over the **bad set**. Let $y = X(s,t,x,v)$, $w = V(s,t,x,v)$. By Preliminary 2(B), 3(B),

$$\frac{1}{2}P < \frac{1}{2}|v| < |w| < 2|v|,$$
$$\frac{1}{2}P < \frac{1}{2}|v - \hat{U}(t)| < |w - \hat{U}(s)| < 2|v - \hat{U}(t)|$$

and either

$$|y - \hat{X}(s)| < R|v|^{-3} < 8R|w|^{-3}$$

or

$$|y - \hat{X}(s)| < R|v - \hat{U}(t)|^{-3} < 8R|w - \hat{U}(s)|^{-3}.$$

Therefore

$$\begin{aligned}
\iiint_B &= \iiint_B \frac{f(t,x,v)\,dv\,dx\,ds}{|X(s,t,x,v) - \hat{X}(s)|^2} = \iiint_B \frac{f(s,y,w)}{|y - \hat{X}(s)|^2}\,dw\,dy\,ds \\
&\le \int_{t-\Delta}^{t} \int_{\frac{1}{2}P<|w|<Q(t)} \int_{|y-\hat{X}(s)|<8R|w|^{-3}} \frac{f(s,y,w)}{|y - \hat{X}(s)|^2}\,dw\,dy\,ds \\
&\quad + \int_{t-\Delta}^{t} \int_{\frac{1}{2}P<|w-\hat{U}(s)|<2Q(t)} \int_{|y-\hat{X}(s)|<8R|w-\hat{U}(s)|^{-3}} \frac{f(s,y,w)\,dw\,dy\,ds}{|y - \hat{X}(s)|^2} \qquad (4.40) \\
&\le c \int_{t-\Delta}^{t} \int_{\frac{1}{2}P<|w|<Q(t)} 4\pi \cdot 8R|w|^{-3}\,dw\,ds \\
&\quad + c \int_{t-\Delta}^{t} \int_{\frac{1}{2}P<|w-\hat{U}(s)|<2Q(t)} 4\pi \cdot 8R|w - \hat{U}(s)|^{-3}\,dw\,ds \\
&\le cR\Delta \cdot \ln\left(\frac{4Q(t)}{P}\right).
\end{aligned}$$

This is the bound for the **bad set**.

Now consider estimation of the integral over the **ugly set**. The time integral smoothing, a *crucial issue*, will be used here. Let $Z(s) = X(s,t,x,v) - \hat{X}(s)$; pick $s_0 \in [t - \Delta, t]$ such that $|Z(s_0)|$ is minimal. Now

$$|\dot{Z}(s)| = |V(s,t,x,v) - \hat{U}(s)| > \frac{1}{2}|v - \hat{U}(t)| \qquad (4.41)$$

by Preliminary 3(B). Set

$$\overline{Z}(s) = Z(s_0) + \dot{Z}(s_0)(s - s_0). \qquad (4.42)$$

Now we claim that

$$(s - s_0)Z(s_0) \cdot \dot{Z}(s_0) \ge 0 \quad \text{on} \quad t - \Delta \le s \le t.$$

Using

$$\frac{1}{2}\frac{d}{ds}|Z(s)|^2 = Z(s)\cdot\dot{Z}(s)$$

we see that there are three cases:

(i) if $s_0 = t-\Delta$, then $s - s_0 \ge 0$ and $Z(s_0)\cdot\dot{Z}(s_0) = \frac{1}{2}\frac{d}{ds}|Z(s)|^2 \ge 0$ because the minimum occurs at the left end point.

(ii) if $s_0 = t$, $s - s_0 \le 0$ and $Z(s_0)\cdot\dot{Z}(s_0) \le 0$ for similar reasons.

(iii) if $t-\Delta < s_0 < t$, then $Z(s_0)\cdot\dot{Z}(s_0) = 0$.

This establishes the claim. Thus

$$\begin{aligned} |\overline{Z}(s)|^2 &\ge |Z(s_0)|^2 + |\dot{Z}(s_0)(s-s_0)|^2 \\ &\ge \frac{|v-\hat{U}(t)|^2}{4}(s-s_0)^2. \end{aligned} \tag{4.43}$$

Now it follows from Taylor's Theorem that

$$|\ddot{Z}(s) - \ddot{\overline{Z}}(s)| \le 2c_1\, Q^{4/3}(t)$$

so that

$$\begin{aligned} |Z(s) - \overline{Z}(s)| &\le c_1 Q^{4/3}(t)(s-s_0)^2 \\ &\le c_1 Q^{4/3}(t)\Delta|s-s_0| \\ &\le \frac{P}{4}|s-s_0| \\ &< \frac{|v-\hat{U}(t)|}{4}|s-s_0|. \end{aligned} \tag{4.44}$$

Therefore

$$\begin{aligned} |Z(s)| &\ge |\overline{Z}(s)| - |Z(s)-\overline{Z}(s)| \\ &\ge \frac{|v-\hat{U}(t)|}{2}|s-s_0| - \frac{|v-\hat{U}(t)|}{4}|s-s_0| \\ &= \frac{|v-\hat{U}(t)|}{4}|s-s_0|. \end{aligned} \tag{4.45}$$

Now fix v and define

$$\sigma_1(r) = \begin{cases} r^{-1} & r \ge (R|v|^{-3})^2 \\ (R|v|^{-3})^{-2} & \text{else;} \end{cases}$$

$$\sigma_2(r) = \begin{cases} r^{-1} & r \ge (R|v-\hat{U}(t)|^{-3})^2 \\ (R|v-\hat{U}(t)|^{-3})^{-2} & \text{else.} \end{cases}$$

Each is non-increasing.

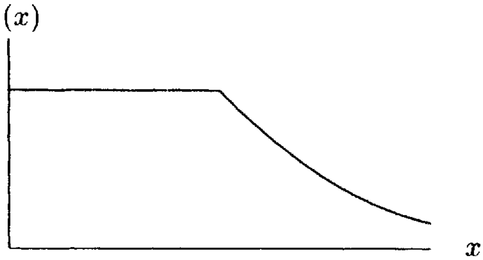

Fig. 4.1 *The Upper Bounds* $\sigma_i(x)$

Then for $i = 1, 2$,

$$|Z(s)|^{-2}\chi_U(s,x,v) \le \sigma_i(|Z(s)|^2) \le \sigma_i\left(\frac{|v-\hat{U}(t)|^2|s-s_0|^2}{16}\right). \tag{4.46}$$

Hence

$$\int_{t-\Delta}^{t} |Z(s)|^{-2}\chi_U(s,x,v)\,ds \le \int_{-\infty}^{\infty} \sigma_i\left(\left(\frac{|v-\hat{U}(t)||s-s_0|}{4}\right)^2\right) ds \tag{4.47}$$
$$= 2\int_0^\infty \sigma_i\left(\left(\frac{|v-\hat{U}(t)|r}{4}\right)^2\right) dr$$
$$= 8\int_0^\infty \frac{\sigma_i(\eta^2)\,d\eta}{|v-\hat{U}(t)|}.$$

Now

$$\int_0^\infty \sigma_1(\eta^2)\,d\eta = \int_0^{R|v|^{-3}} (R|v|^{-3})^{-2}\,d\eta + \int_{R|v|^{-3}}^\infty \eta^{-2}\,d\eta \tag{4.48}$$
$$= 2(R|v|^{-3})^{-1}.$$

Similarly we treat $\int_0^\infty \sigma_2(\eta^2)\,d\eta$. Therefore

$$\int_{t-\Delta}^{t} |Z(s)|^{-2}\chi_U(s,x,v)\,ds \le \begin{cases} 16|v-\hat{U}(t)|^{-1}(R|v|^{-3})^{-1}, & i=1 \\ 16|v-\hat{U}(t)|^{-1}(R|v-\hat{U}(t)|^{-3})^{-1}, & i=2. \end{cases}$$

Hence

$$\int_{t-\Delta}^{t} |Z(s)|^{-2}\chi_U(s,x,v)\,ds \le \frac{16|v-\hat{U}(t)|^{-1}}{R}\cdot\Big(\min\{|v|,|v-\hat{U}(t)|\}\Big)^3$$
$$\le 16R^{-1}|v|^2.$$

It follows that

$$\iiint_U \le cR^{-1}\iint f(t,x,v)|v|^2\,dv\,dx \le cR^{-1} \tag{4.49}$$

by the energy bound (4.16).

Now we collect the estimates (4.39), (4.40), (4.49) to get

$$\begin{aligned}(4.50)\qquad \frac{1}{\Delta}\int_{t-\Delta}^{t}|E(s,\hat{X}(s))|\,ds &\le c\left(P^{4/3}+R\ln\left(\frac{4Q(t)}{P}\right)+\frac{1}{R\Delta}\right)\\ &\le c\left(P^{4/3}+R\ln\left(\frac{4Q(t)}{P}\right)+R^{-1}\cdot P^{-1}Q^{4/3}(t)\right).\end{aligned}$$

We take $P=Q^{4/11}(t)$; $R=Q^{16/33}(t)\ \ln^{-1/2}\left(\frac{4Q(t)}{P}\right)$. Then

$$\begin{aligned}\frac{1}{\Delta}\int_{t-\Delta}^{t}|E(s,\hat{X}(s))|\,ds &\le c\Bigg(Q(t)^{16/33}+Q(t)^{16/33}\cdot\ln^{1/2}\left(\frac{4Q(t)}{Q^{4/11}(t)}\right)\\ (4.51)\qquad &\qquad +Q(t)^{4/3-4/11-16/33}\ln^{1/2}\left(\frac{4Q(t)}{Q^{4/11}(t)}\right)\Bigg)\\ &\le cQ(t)^{16/33}\ln^{1/2}Q(t).\end{aligned}$$

4.5. The Bound on the Velocity Support

The bound on $Q(t)$ is obtained as follows. From (4.3) and (4.51) we have

$$\begin{aligned}(4.52)\qquad |\hat{U}(t)| &\le |\hat{U}(t-\Delta)|+\int_{t-\Delta}^{t}|E(s,\hat{X}(s))|\,ds\\ &\le Q(t-\Delta)+c\Delta Q^{16/33}(t)\ln^{1/2}Q(t).\end{aligned}$$

Here c is independent of the particular characteristic $(\hat{X},\hat{U})$, so

$$(4.53)\qquad Q(t)\le Q(t-\Delta)+c\Delta Q^{16/33}(t)\ln^{1/2}Q(t)$$

for

$$\begin{aligned}\Delta &= \min\left\{t,\frac{1}{4c_1}Q^{-4/3}(t)\cdot Q^{4/11}(t)\right\}\\ &= \min\left\{t,\frac{1}{4c_1}Q(t)^{-32/33}\right\}.\end{aligned}$$

Since Q is non–decreasing, there exists T_1 such that

$$(4.54)\qquad \Delta=\begin{cases} t & t\le T_1\\ \frac{1}{4c_1}Q(t)^{-32/33} & t>T_1.\end{cases}$$

Take t_0 in the interval of existence. Without loss of generality, $t_0>T_1$. Let

$$\begin{aligned}t_1 &= t_0-\frac{1}{4c_1}Q(t_0)^{-32/33},\\ t_{i+1} &= t_i-\frac{(Q(t_i))^{-32/33}}{4c_1}\qquad (i=1,2,\ldots)\end{aligned}$$

as long as $t_i > T_1$. Then

$$t_i - t_{i+1} = \frac{1}{4c_1} Q^{-32/33}(t_i) \geq \frac{1}{4c_1} Q^{-32/33}(t_0) \tag{4.55}$$

which is a uniform lower bound on the length of each subinterval. So there is a first i, say $i = k$, such that $t_k \leq T_1$. Thus $t_k \geq 0$ and therefore

$$(t_0 - t_1) + (t_1 - t_2) + \cdots + (t_{k-1} - t_k) \geq k \cdot \frac{Q(t_0)^{-32/33}}{4c_1} \tag{4.56}$$

which implies that

$$Q(t_0)^{-32/33} \cdot k \leq 4c_1 t_0. \tag{4.57}$$

Now we have

$$\begin{aligned} Q(t_0) &= Q(t_k) + \sum_{i=0}^{k-1} [Q(t_i) - Q(t_{i+1})] \\ &\leq Q(t_k) + c \sum_{i=0}^{k-1} \Delta \cdot Q^{16/33}(t_i) \ln^{1/2} Q(t_i) \\ &\leq Q(T_1) + c \sum_{i=0}^{k-1} \frac{1}{4c_1} Q^{-32/33}(t_0) \cdot Q^{16/33}(t_i) \ln^{1/2} Q(t_i) \\ &\leq Q(T_1) + c \cdot k Q^{-32/33}(t_0) \cdot Q^{16/33}(t_0) \ln^{1/2} Q(t_0) \\ &\leq Q(T_1) + c t_0 \cdot Q^{16/33}(t_0) \ln^{1/2} Q(t_0). \end{aligned} \tag{4.58}$$

Therefore $Q(t_0)$ is bounded, and the proof is complete.

We conclude this proof with some remarks on related techniques and problems. The proof just given is a modification of Schaeffer's simplification [41] of Pfaffelmoser's original proof [34]. In the proof of Horst [28] the integral of E along characteristics is partitioned in a rather similar fashion, but Horst shows that the growth rate of $Q(t)$ is essentially first-order in t, as mentioned in the beginning of this chapter, and he allows more general data. A completely different and elegant argument is given by Lions and Perthame in [31]. They show that higher moments of f in v can be estimated. They employ a representation for ρ which is obtained by integrating along the straight line characteristics of the Vlasov equation (4.1).

The *relativistic* Vlasov–Poisson system has the form

$$f_t + \hat{v} \cdot \nabla_x f + \gamma E \cdot \nabla_v f = 0$$

where E and ρ are given as before, $\gamma = \pm 1$ and

$$\hat{v} = \frac{v}{\sqrt{1 + |v|^2}}.$$

This is not a physically well-posed equation, since it lacks Lorentz invariance (the proper coupling to the Maxwell equations restores the desired invariance). Nevertheless, it seems that this should be mathematically simpler and nearly a prerequisite for solving the Vlasov–Maxwell system globally in time. However, none of the now known methods applies. Thus the global solvability for "large" data remains open. In this case, the sign of the nonlinear term is important; finite–time blowup can occur for "large" data if the force is "attractive", as we show below (cf. [21]).

4.6. Blow–up in the Gravitational Case

We conclude this chapter with a brief discussion of the gravitational problem for the Vlasov–Poisson system. In this case the energy has an indefinite sign, which allows the possibility of finite–time blow–up in higher dimension.

We begin with the argument of Horst ([25]) which shows that if we pose the non–relativistic problem in a phase space $\mathbb{R}^n \times \mathbb{R}^n$ with $n \geq 4$, then any smooth solution can exist only on a *finite* interval of time.

The equation can be written as

$$
\begin{aligned}
\text{(4.59)}\qquad & f_t + v \cdot \nabla_x f - E \cdot \nabla_v f = 0 \quad (x, v \in \mathbb{R}^n);\\
& \rho(t,x) = \int f \, dv,\\
& E(t,x) = \nabla_x u, \quad \Delta u = \rho,\\
& f(0,x,v) = f_0(x,v).
\end{aligned}
$$

The conserved energy in this case is

$$\iint |v|^2 f \, dv \, dx - \int |E|^2 \, dx = \text{const.} \equiv \mathcal{E}_0.$$

We will assume that $\mathcal{E}_0 < 0$, and that $n \geq 4$. Let $f(t,x,v)$ be a nonnegative smooth solution with data of compact support existing on $[0,T) \times \mathbb{R}^n \times \mathbb{R}^n$ with finite energy. Then we claim that $T < \infty$.

For the proof we put $r = |x|$ and compute

$$
\begin{aligned}
\frac{d}{dt} \iint r^2 f \, dx \, dv &= \iint r^2 \left(-\nabla_x \cdot (vf) + \nabla_v \cdot (Ef)\right) dv \, dx\\
&= 2 \iint (x \cdot v) f \, dv \, dx
\end{aligned}
$$

and hence

$$
\begin{aligned}
\text{(4.60)}\qquad \frac{d^2}{dt^2} \iint r^2 f \, dx \, dv &= 2 \iint x \cdot v \left(-\nabla_x \cdot (vf) + \nabla_v \cdot (Ef)\right) dv \, dx\\
&= 2 \iint |v|^2 f \, dx \, dv - 2 \iint x \cdot E f \, dx \, dv.
\end{aligned}
$$

Now

$$\begin{aligned}\iint x \cdot Ef\,dx\,dv &= \int \rho x \cdot E\,dx \\ &= \int \sum_{k,j} x_j \partial_j u\, \partial_k^2 u\,dx \\ &= -\int \sum_{k,j} \partial_k u\,(\delta_{kj}\partial_j u + x_j \partial_j \partial_k u)\,dx \\ &= -\int |E|^2\,dx - \frac{1}{2}\sum_{k,j}\int x_j \partial_j u_{x_k}^2\,dx \\ &= -\int |E|^2\,dx + \frac{n}{2}\int |E|^2\,dx \\ &= \left(\frac{n}{2}-1\right)\int |E|^2\,dx.\end{aligned}$$

Thus from (4.60) we get

$$\begin{aligned}\frac{1}{2}\frac{d^2}{dt^2}\int r^2 \rho\,dx &= \iint |v|^2 f\,dv\,dx - \int \rho x \cdot E\,dx \\ &= \iint |v|^2 f\,dv\,dx - \left(\frac{n}{2}-1\right)\int |E|^2\,dx \\ &= \mathcal{E}_0 + \left(2 - \frac{n}{2}\right)\int |E|^2\,dx \\ &= \mathcal{E}_0 + \left(\frac{4-n}{2}\right)\int |E|^2\,dx \\ &\leq \mathcal{E}_0.\end{aligned}$$

Integrating twice in time, we have

$$\int r^2 \rho(t,x)\,dx \leq \int r^2 \rho(0,x)\,dx + t\int r^2 \rho_t(0,x)\,dx + \mathcal{E}_0 t^2.$$

The right-hand side here is negative for sufficiently large t (because of the assumption $\mathcal{E}_0 < 0$) while the left-hand side remains nonnegative. This leads to an upper bound for T, and establishes the result.

Lastly, consider the relativistic Vlasov-Poisson system with $\gamma = -1$ in three space dimensions:

$$f_t + \hat{v} \cdot \nabla_x f - E \cdot \nabla_v f = 0$$

where $E = \nabla_x u$, $\Delta u = \rho = \int f\,dv$ and

$$\hat{v} = \frac{v}{\sqrt{1+|v|^2}}.$$

The conserved energy for this problem is

$$\int\int \sqrt{1+|v|^2} f \, dv \, dx - \frac{1}{2}\int |E|^2 \, dx = \text{const.} \equiv \mathcal{E}_1.$$

We will assume that $\mathcal{E}_1 < 0$. Let $f(t,x,v)$ be a nonnegative *spherically symmetric* smooth solution with data of compact support existing on $[0,T) \times \mathbb{R}^3 \times \mathbb{R}^3$ with finite energy. Then we again have the result that $T < \infty$.

Using direct calculations very similar to those above, we obtain the *dilation identity*

$$\frac{d}{dt}\int\int x \cdot vf \, dx \, dv = \mathcal{E}_1 - \int\int \frac{f \, dv \, dx}{\sqrt{1+|v|^2}}$$

and, with $r = |x|$,

$$\frac{d}{dt}\int\int r^2\sqrt{1+|v|^2} f \, dx \, dv = 2\int\int x \cdot vf \, dx \, dv - \int r^2 E \cdot j \, dx \tag{4.61}$$

where

$$j = \int \hat{v} f \, dv.$$

Integrating the dilation identity once in time, we have

$$\int\int x \cdot vf(t,x,v) \, dx \, dv \le \int\int x \cdot vf(0,x,v) \, dx \, dv + \mathcal{E}_1 t. \tag{4.62}$$

Now we use the radial nature of the solution. As is well known, the spherically symmetric form of the solution to the Poisson equation $\Delta u = \rho$ is

$$u(t,r) = -\frac{1}{r}\int_0^r \lambda^2 \rho(t,\lambda) \, d\lambda - \int_r^\infty \lambda \rho(t,\lambda) \, d\lambda$$

so that

$$E(t,x) = \frac{x}{r^3}\int_0^r \lambda^2 \rho(t,\lambda) \, d\lambda. \tag{4.63}$$

Set

$$M(t,r) = \frac{x}{r}\int_0^r \lambda^2 \rho(t,\lambda) \, d\lambda.$$

Then we have a uniform in time bound $|M(t,r)| \le M_0 < \infty$ by the L^1 conservation law. Thus

$$\left|\int r^2 E \cdot j \, dx\right| \le M_0 \int |j| \, dx \le M_0 \int \rho \, dx \le M_0^2. \tag{4.64}$$

Hence from (4.61) we have

$$\begin{aligned}\frac{d}{dt}\int\int r^2\sqrt{1+|v|^2} f \, dx \, dv &= 2\int\int x \cdot vf \, dx \, dv - \int r^2 E \cdot j \, dx \\ &\le 2\left(\int\int x \cdot vf(0,x,v) \, dx \, dv + \mathcal{E}_1 t\right) + M_0^2\end{aligned}$$

where we have used (4.62). Integrating this once in time, we get

$$\begin{aligned}\int\!\!\int r^2\sqrt{1+|v|^2}f(t,x,v)\,dx\,dv &\le \int\!\!\int r^2\sqrt{1+|v|^2}f(0,x,v)\,dx\,dv \\ &\quad + \left(2\int\!\!\int x\cdot vf(0,x,v)\,dx\,dv + M_0^2\right)t + \mathcal{E}_1 t^2 \\ &\equiv c_1 + c_2 t + \mathcal{E}_1 t^2.\end{aligned}$$

Once again we find an upper bound for T using the hypothesis that $\mathcal{E}_1 < 0$, and the proof is complete.

In the references we include a number of papers on related problems which have not been discussed. In particular, there one will find results on stability of certain solutions to the Vlasov–Poisson system, numerical algorithms and treatments of the Vlasov–Poisson–Fokker–Planck system (which models collisions using diffusion in v–space).

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Chapter 5

THE VLASOV–MAXWELL SYSTEM

5.1. Collisionless Plasmas

The goal in this chapter is to derive a sufficient condition for the global existence of a smooth solution to the relativistic Vlasov–Maxwell System (cf. [31]). As in the Vlasov–Poisson case, this condition reduces to the control of large velocities.

A **Plasma** is a completely ionized gas. We assume as usual two species: electrons, with charge $-e$, and positive ions, with charge Ze $(Z \in \mathbb{N})$. The major assumptions are that the plasma is at high temperature, is of low density, and that collisions are unimportant. "High temperature" means

$$T \gg \frac{e^2}{\bar{r}} \cong e^2 N^{1/3}$$

where T is the temperature, N is the total number of particles per unit volume, and $\bar{r} \sim N^{-1/3}$ is the mean distance between them. The **Debye length** a is defined by

$$a^{-2} = \frac{4\pi}{T} \sum_{\alpha} N_\alpha (Z_\alpha e)^2 \qquad \begin{pmatrix} \text{sum over all} \\ \text{types of ions} \end{pmatrix}.$$

Then a is the distance at which the Coulomb field of a charge in the plasma is screened. If we have only one type of ion, with $Z = 1$, then

$$\frac{1}{a^2} = \frac{4\pi}{T} N (Ze)^2, \text{ or } a = \left(\frac{T}{4\pi N e^2} \right)^{1/2} \tag{5.1}$$

so the above condition is

$$\frac{e^2 N^{1/3}}{T} \ll 1, \quad \text{i.e.,} \quad \frac{e^2 N^{1/3}}{4\pi N e^2 a^2} \ll 1, \quad \text{or} \quad \frac{\bar{r}^2}{4\pi a^2} \ll 1. \tag{5.2}$$

This can be interpreted as saying the mean distance between particles is small with respect to the Debye length i.e., the ion cloud around a charge must contain many particles.

We can consider a plasma **collisionless** when the *effective collision frequency* ν (= the reciprocal of the mean free time of a particle) $\ll w =$ *frequency of variation of* E, B. Under these circumstances

$$\frac{\partial f}{\partial t} \gg \text{collision term.}$$

Another condition under which the collisionless approximation makes sense is this: let

$$\begin{aligned} \ell &= \text{particle mean free path} \cong \frac{\overline{v}}{\nu} \\ L &= \text{distance over which the field varies (= ``field wavelength'').} \end{aligned}$$

Then if $\nu \ll \frac{\overline{v}}{L}$, the streaming term is dominant:

$$v \cdot \nabla_x f \gg \text{collision term.}$$

These conclusions are established in [20].

5.2. Control of Large Velocities

We consider the relativistic case. Assume that we have several species with masses m_α and charges e_α, $1 \leq \alpha \leq N$. The *relativistic velocity* is

$$\hat{v}_\alpha = \frac{v}{\sqrt{m_\alpha^2 + |v|^2/c^2}}$$

where c is the speed of light. Thus $|\hat{v}_\alpha| < c$ for each α.

The particle densities $f_\alpha(t, x, v)$ satisfy

$$\partial_t f_\alpha + \hat{v}_\alpha \cdot \nabla_x f_\alpha + e_\alpha \left(E + \frac{\hat{v}_\alpha}{c} \times B \right) \cdot \nabla_v f_\alpha = 0, \tag{5.3}$$

$$\rho = 4\pi \int \sum_\alpha e_\alpha f_\alpha \, dv, \qquad j = 4\pi \int \sum_\alpha \hat{v}_\alpha e_\alpha f_\alpha \, dv,$$

$$\begin{aligned} \partial_t E &= c\nabla \times B - j \qquad & \nabla \cdot E &= \rho \\ \partial_t B &= -c\nabla \times E & \nabla \cdot B &= 0. \end{aligned}$$

Given are $f_\alpha(0, x, v) = f_{\alpha 0}(x, v)$, $E(0, x) = E_0(x)$, $B(0, x) = B_0(x)$ satisfying $\nabla \cdot E_0 = \rho_0$, $\nabla \cdot B_0 = 0$, $\int \rho_0 \, dx = 0$.

THEOREM 5.2.1 *Let* $0 \leq f_{\alpha 0} \in C_0^1$; $E_0, B_0 \in C^2$ *satisfy the above constraints. Assume there exists a continuous function* $\beta(t)$ *such that for all* x, α, $f_\alpha(t, x, v) = 0$ *for* $|v| > \beta(t)$. *Then there exists a unique* C^1 *solution for all* t.

The proof in [9] is based on a representation for the fields which eliminates the loss of derivatives suffered by solutions to the Maxwell System. We simplify the exposition by dropping the π factors and taking $c = 1$, $N = 1$, $m_\alpha = e_\alpha = 1$. Then the system appears as

$$
\begin{aligned}
&\partial_t f + \hat{v}\cdot\nabla_x f + (E + \hat{v}\times B)\cdot\nabla_v f = 0, \\
&\rho = \int f\,dv, \qquad j = \int \hat{v} f\,dv, \\
&\partial_t E = \nabla\times B - j, \qquad \nabla\cdot E = \rho \\
&\partial_t B = -\nabla\times E, \qquad \nabla\cdot B = 0.
\end{aligned} \tag{5.4}
$$

Traditionally one writes $K = E + \hat{v}\times B$ for the force.

f is constant on characteristics

$$\dot{x} = \hat{v}, \qquad \dot{v} = K$$

whose solutions we write as

$$X(s,t,x,v),\ V(s,t,x,v) \quad \text{with} \quad X(t,t,x,v) = x,\ V(t,t,x,v) = v.$$

Hence

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v))$$

and so f remains nonnegative and bounded, provided $0 \le f_0 \le \max f_0 < \infty$.

5.3. Representation of the Fields

THEOREM 5.3.1 *Let $\beta(t)$ exist as in the hypotheses of Theorem 5.2.1. Call $S = \partial_t + \sum_{k=1}^3 \hat{v}_k\partial_{x_k}$. Then for $i = 1,2,3$, the fields admit the representations*

$$
\begin{aligned}
4\pi E^i(t,x) &= (E^i)_0(t,x) + E^i_T(t,x) + E^i_S(t,x), \\
4\pi B^i(t,x) &= (B^i)_0(t,x) + B^i_T(t,x) + B^i_S(t,x) \qquad \textit{where} \\
E^i_T(t,x) &= -\int_{|y-x|\le t}\int \frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2} f(t-|y-x|,y,v)\,dv\frac{dy}{|y-x|^2}; \\
E^i_S(t,x) &= -\int_{|y-x|\le t}\int \frac{(\omega_i+\hat{v}_i)}{1+\hat{v}\cdot\omega}(Sf)(t-|x-y|,y,v)\,dv\frac{dy}{|y-x|}.
\end{aligned}
$$

For the fields B^i there is a similar representation; we need only replace $\omega_i + \hat{v}_i$ in each expression above by $(\omega\times\hat{v})_i$. Here $\omega = \frac{y-x}{|y-x|}$.

Proof. We put $r = |y-x|$. Note that $\partial_{y_i}[f(t-|x-y|,y,v)] = \partial_{y_i} f - \omega_i\partial_t f \equiv T_i f$, and that any T_i is a tangential derivative along the surface of a backward characteristic cone. The idea is to replace the usual operators ∂_t, ∂_i by T_i, S. We can invert these:

$$
\begin{aligned}
\partial_t &= \frac{S - \hat{v}\cdot T}{1+\hat{v}\cdot\omega}, \\
\partial_i &= T_i + \frac{\omega_i}{1+\hat{v}\cdot\omega}(S - \hat{v}\cdot T) \\
&= \frac{\omega_i S}{1+\hat{v}\cdot\omega} + \left(\delta_{ij} - \frac{\omega_i\hat{v}_j}{1+\hat{v}\cdot\omega}\right)T_j.
\end{aligned} \tag{5.5}
$$

The summation convention is followed here. Now compute

$$(\partial_t^2 - \Delta)E^i = -(\partial_i \rho + \partial_t j^i) = -\int (\partial_i f + \hat{v}_i \partial_t f)\, dv. \tag{5.6}$$

In the integrand we have

$$\partial_i + \hat{v}_i \partial_t = \frac{(\omega_i + \hat{v}_i)S}{1 + \hat{v}\cdot\omega} + \left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1 + \hat{v}\cdot\omega}\right) T_j. \tag{5.7}$$

Therefore

$$\begin{aligned} 4\pi E^i(t,x) &= \underbrace{(E^i)_0(t,x)}_{\text{data}} \\ &\quad - \int_{|y-x|\le t}\int \frac{(\omega_i + \hat{v}_i)}{1+\hat{v}\cdot\omega}(Sf)(t - |y-x|, y, v)\, dv \frac{dy}{|y-x|} \\ &\quad - \int_{|y-x|\le t}\int \underbrace{\left(\delta_{ij} - \frac{(\omega_i + \hat{v}_i)\hat{v}_j}{1+\hat{v}\cdot\omega}\right)}_{a_j}(T_j f)(t - |x-y|, y, v)\, dv \frac{dy}{|y-x|}. \end{aligned}$$

The middle term has been called E^i_S. The last term is by integration by parts

$$\begin{aligned} &-\int_{|y-x|\le t}\int \frac{a_j}{r}\frac{\partial}{\partial y_j}[f(t - |x-y|, y, v)]\, dy\, dv \\ &= \underbrace{-\int_{|y-x|=t}\int \frac{\omega_j a_j}{r} f(0,y,v)\, dS_y\, dv}_{\text{part of } (E^i)_0} + \int_{|y-x|\le t}\int f \frac{\partial}{\partial y_j}\frac{a_j}{r}\, dy\, dv. \end{aligned} \tag{5.8}$$

Now a lengthy but elementary calculation gives

$$\partial_j\left[\frac{a_j}{r}\right] = \frac{(\omega_i + \hat{v}_i)(|\hat{v}|^2 - 1)}{r^2(1 + \hat{v}\cdot\omega)^2}. \tag{5.9}$$

The calculation for the B field is similar since

$$(\partial_t^2 - \Delta)B^1 = (\nabla \times j)_1 = \int (\partial_2 \hat{v}_3 f - \partial_3 \hat{v}_2 f)\, dv. \tag{5.10}$$

This proves Theorem 5.3.1

Proof of Uniqueness. Let $(f^{(1)}, E^{(1)}, B^{(1)})$ and $(f^{(2)}, E^{(2)}, B^{(2)})$ be two classical solutions of (5.4) with the same Cauchy data. Define

$$f = f^{(1)} - f^{(2)},\ E = E^{(1)} - E^{(2)},\ B = B^{(1)} - B^{(2)},\ K^{(i)} = E^{(i)} + \hat{v}\times B^{(i)}$$
$$K = K^{(1)} - K^{(2)} \quad (i = 1,2).$$

Subtracting the equations, we have (5.6), (5.10), $\rho = \int f\,dv,\ j = \int \hat{v} f\,dv$ and

$$\begin{aligned}(\partial_t + \hat{v}\cdot\nabla_x)f \equiv Sf &= \nabla_v \cdot \{-K^{(1)}f^{(1)} + K^{(2)}f^{(2)}\}\\ &= \nabla_v \cdot \{-Kf^{(1)} - K^{(2)}f\}.\end{aligned}$$

Using the Representation Theorem 5.3.1, we can write

$$4\pi E(t,x) = E_T(t,x) + E_S(t,x) \quad \text{where}$$

$$E_T(t,x) = -\iint\limits_{|y-x|\le t} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2} f(t-|y-x|,y,v)\,dv\,\frac{dy}{|y-x|^2},$$

$$E_S(t,x) = -\iint\limits_{|y-x|\le t} \nabla_v\left\{\frac{(\omega+\hat{v})}{1+\hat{v}\cdot\omega}\right\}\cdot(Kf^{(1)} + K^{(2)}f)\,dv\,\frac{dy}{|y-x|}.$$

Here in the E_S term we have integrated by parts in v, using the fact that Sf is a pure v divergence. A similar representation holds for B. f has compact support in v, so the expression $1+\hat{v}\cdot\omega$ is bounded away from 0. Moreover, the fields are bounded by hypothesis. Adding these, estimating, and using the support property, we get for $t \le T$, with $|\cdot|_0$ denoting the maximum norm,

$$\begin{aligned}|E(t)|_0 + |B(t)|_0 \le C_T \int_0^t \Big(&|f(\tau)|_0 + \{|E^{(2)}(\tau)|_0 + |B^{(2)}(\tau)|_0\}|f(\tau)|_0\\ &+ (|E(\tau)|_0 + |B(\tau)|_0)|f^{(1)}(\tau)|_0\Big)\,d\tau.\end{aligned}$$

Since $E^{(2)}, B^{(2)}$ and $f^{(1)}$ are bounded, we have on $[0,T]$

$$|E(t)|_0 + |B(t)|_0 \le C_T \int_0^t (|f(\tau)|_0 + |E(\tau)|_0 + |B(\tau)|_0)\,d\tau. \tag{5.11}$$

On the other hand, the equation for f above can be written as

$$\partial_t f + \hat{v}\cdot\nabla_x f + K^{(1)}\cdot\nabla_v f = -K\cdot\nabla_v f^{(2)}.$$

Consider the characteristics of this equation, defined by the solutions of

$$\dot{x} = \hat{v},\ \dot{v} = K^{(1)},\ \dot{f} = -K\cdot\nabla_v f^{(2)}.$$

Then f can be written as a line integral over such a characteristic curve of the right-hand side $-K\cdot\nabla_v f^{(2)}$. Hence

$$|f(t)|_0 \le C\int_0^t |K\cdot\nabla_v f^{(2)}(\tau)|_0\,d\tau.$$

Since $\nabla_v f^{(2)}$ is bounded,

$$|f(t)|_0 \le C_T \int_0^t |K(\tau)|_0\, d\tau \le C_T \int_0^t (|E(\tau)|_0 + |B(\tau)|_0)\, d\tau. \tag{5.12}$$

We add (5.11) and (5.12) and apply the Gronwall lemma to conclude that $E = B = f = 0$, as desired. This proves the uniqueness assertion in Theorem 5.2.1.

5.4. Representation of the Derivatives of the Fields

THEOREM 5.4.1 *Let $\beta(t)$ exist as in the hypotheses of Theorem 5.2.1. Then for i, $k = 1, 2, 3$,*

$$\begin{aligned}\partial_k E^i = \underbrace{(\partial_k E^i)_0}_{data} &+ \int_{r\le t}\int a(\omega,\hat v) f\, dv \frac{dy}{r^3} + \int_{|\omega|=1}\int d(\omega,\hat v) f(t,x,v)\, d\omega\, dv \\ &+ \int_{r\le t}\int b(\omega,\hat v)(Sf)\, dv\frac{dy}{r^2} + \int_{r\le t} c(\omega,\hat v) S^2 f\, dv \frac{dy}{r}.\end{aligned}$$

Here $f, Sf, S^2 f$ without explicit arguments are evaluated at $(t - |x-y|, y, v)$, and $r = |y - x|$. The functions a, b, c, d are C^∞ except at $1 + \hat v\cdot\omega = 0$ and have algebraic singularities at such points. Moreover, $\int_{|\omega|=1} a(\omega,\hat v)\, d\omega = 0$. Hence the apparently singular integral above (containing the factor $|y-x|^{-3}$) is in fact convergent for suitably smooth f. There is a similar representation for $\partial_k B^i$.

Proof. We put $z = x - y$ in the field representation of Theorem 5.3.1 and then apply $\partial/\partial x_k$. There results

$$\begin{aligned}4\pi\partial_k E^i = (\partial_k E^i)_0 &- \iint\limits_{|y-x|\le t} \frac{(\omega_i+\hat v_i)(1-|\hat v|^2)}{(1+\hat v\cdot\omega)^2}\partial_k f(t-|x-y|,y,v)\, dv \frac{dy}{|y-x|^2} \\ &- \iint\limits_{|y-x|\le t} \frac{(\omega_i+\hat v_i)}{1+\hat v\cdot\omega}\partial_k(Sf)(t-|x-y|,y,v)\, dv\frac{dy}{|y-x|} \\ = (\partial_k E^i)_0 &- \iint\limits_{|y-x|\le t} \frac{(\omega_i+\hat v_i)(1-|\hat v|^2)}{(1+\hat v\cdot\omega)^2}\left[\left(\delta_{jk} - \frac{\omega_k\hat v_j}{1+\hat v\cdot\omega}\right)T_j + \frac{\omega_k}{1+\hat v\cdot\omega}S\right] f\, dv\frac{dy}{r^2} \\ &- \iint\limits_{|y-x|\le t} \frac{(\omega_i+\hat v_i)}{1+\hat v\cdot\omega}\left[\left(\delta_{jk} - \frac{\omega_k\hat v_j}{1+\hat v\cdot\omega}\right)T_j + \frac{\omega_k}{1+\hat v\cdot\omega}S\right](Sf)\, dv\frac{dy}{r}.\end{aligned} \tag{5.13}$$

In the last term we integrate by parts in y, using the fact that T_j is a perfect y_j derivative. Thus the last term is

$$\iint\limits_{|y-x|\le t} \partial_j\left[\frac{(\omega_i+\hat v_i)}{1+\hat v\cdot\omega}\left(\delta_{jk} - \frac{\omega_k \hat v_j}{1+\hat v\cdot\omega}\right)\cdot\frac{1}{r}\right](Sf)(t-|x-y|,y,v)\, dv\, dy \tag{5.14}$$

$$
\begin{aligned}
&- \iint_{|y-x|\le t} \frac{(\omega_i + \hat{v}_i)\omega_k}{r(1+\hat{v}\cdot\omega)^2} S^2 f(t-|x-y|,y,v)\,dv\,dy \\
&- \frac{1}{t}\int_{|y-x|=t}\int \frac{\omega_k(\omega_i+\hat{v}_i)}{(1+\hat{v}\cdot\omega)^2} Sf(0,y,v)\,dv\,dS_y
\end{aligned}
$$

so $c(\omega,\hat{v})$ is the kernel multiplying $S^2 f$. The last data term is lumped into $(\partial_k E^i)_0$. Part of $b(\omega,\hat{v})$ is visible. The other part of $b(\omega,\hat{v})$ comes from the S-part of the second term. The most singular term is the T_j-term in the first expression; it is

(5.15)

$$
E_{TT} = -\lim_{\epsilon\to 0} \iint_{\epsilon\le|y-x|\le t} \frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{r^2(1+\hat{v}\cdot\omega)^2}\left(\delta_{jk} - \frac{\omega_k \hat{v}_j}{1+\hat{v}\cdot\omega}\right) T_j f(t-|x-y|,y,v)\,dv\,dy.
$$

The integral in E_{TT} equals

$$
\begin{aligned}
&\iint_{|y-x|=t} \frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{t^2(1+\hat{v}\cdot\omega)^2}\left(\delta_{jk} - \frac{\omega_k\hat{v}_j}{1+\hat{v}\cdot\omega}\right) f(0,y,v)\,dv\,dS_y \\
&- \iint_{|y-x|=\epsilon} \frac{\omega_j(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{\epsilon^2(1+\hat{v}\cdot\omega)^2}\left(\delta_{jk} - \frac{\omega_k\hat{v}_j}{1+\hat{v}\cdot\omega}\right) f(t-\epsilon,y,v)\,dv\,dS_y \\
&- \iint_{\epsilon\le|y-x|\le t} \partial_j\left[\frac{(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{r^2(1+\hat{v}\cdot\omega)^2}\left(\delta_{jk} - \frac{\omega_k\hat{v}_j}{1+\hat{v}\cdot\omega}\right)\right] f(t-|x-y|,y,v)\,dv\,dy.
\end{aligned}
$$

The first term depends on data only, so is part of $(\partial_k E^i)_0$. The second term converges to

$$
\begin{aligned}
(5.16)\quad & -\int_{|\omega|=1}\int \frac{(\omega_j)(\omega_i+\hat{v}_i)(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2}\left(\delta_{jk} - \frac{\omega_k\hat{v}_j}{1+\hat{v}\cdot\omega}\right) f(t,x,v)\,d\omega\,dv \\
&\equiv -\int_{|\omega|=1}\int d(\omega,\hat{v}) f(t,x,v)\,d\omega\,dv.
\end{aligned}
$$

Since the v-integral is over $|v|\le\beta(t)$, $(1+\hat{v}\cdot\omega)\ge c_T>0$ on $0\le t\le T$. Also, f is pointwise bounded. Therefore the term involving $d(\omega,\hat{v})$ is $0(1)$. (We remark that the integrals which appear here over $|\omega|=1$ can be explicitly computed.) Now in the last term of E_{TT} we have

$$
\begin{aligned}
&\partial_j\left\{\frac{(\omega_i+\hat{v}_i)}{(1+|v|^2)(1+\hat{v}\cdot\omega)^2}\left(\delta_{jk} - \frac{\omega_k\hat{v}_j}{1+\hat{v}\cdot\omega}\right) r^{-2}\right\} \\
&= r^{-3}\cdot\frac{-3(\omega_i+\hat{v}_i)[\omega_k(1-|\hat{v}|^2)+\hat{v}_k(1+\hat{v}\cdot\omega)] + (1+\hat{v}\cdot\omega)^2\delta_{ik}}{(1+|v|^2)(1+\hat{v}\cdot\omega)^4} \\
&\equiv \frac{a(\omega,\hat{v})}{r^3}
\end{aligned}
$$

after a lengthy but elementary computation (cf. [9]).

To show that $\int_{|\omega|=1} a(\omega,\hat{v})\,d\omega = 0$, write

$$a(\omega,\hat{v}) = I + II + III$$

where

$$I = \frac{-3(\omega_i+\hat{v}_i)\omega_k}{(1+|v|^2)^2(1+\hat{v}\cdot\omega)^4}, \tag{5.17}$$
$$II = \frac{-3\hat{v}_k(\omega_i+\hat{v}_i)}{(1+|v|^2)(1+\hat{v}\cdot\omega)^3},$$
$$III = \frac{\delta_{ik}}{[\sqrt{1+|v|^2}+v\cdot\omega]^2}.$$

First we compute

$$J \equiv \int_{|\omega|=1} \frac{d\omega}{[\sqrt{1+|v|^2}+v\cdot\omega]^2} = (1+|v|^2)^{-1}\int_{|\omega|=1}\frac{d\omega}{(1+\hat{v}\cdot\omega)^2} \tag{5.18}$$
$$= 2\pi(1+|v|^2)^{-1}\int_0^\pi \frac{\sin\phi\,d\phi}{(1+|\hat{v}|\cos\phi)^2} = 4\pi.$$

Thus

$$\int_{|\omega|=1} III\ d\omega = 4\pi\delta_{ik}. \tag{5.19}$$

Similarly,

$$\hat{v}_k\int_{|\omega|=1}\frac{d\omega}{[\sqrt{1+|v|^2}+v\cdot\omega]^3} = 4\pi v_k. \tag{5.20}$$

Now

$$I = \frac{-3\omega_k(\omega_i+\hat{v}_i)}{[\sqrt{1+|v|^2}+v\cdot\omega]^4} = \frac{\partial}{\partial v_i}\left[\frac{(\omega_k+\hat{v}_k)-\hat{v}_k}{(\sqrt{1+|v|^2}+v\cdot\omega)^3}\right]$$
$$= -\frac{\partial}{\partial v_i}\left[\frac{\partial}{\partial v_k}\left\{\frac{1}{2}(\sqrt{1+|v|^2}+v\cdot\omega)^{-2}\right\} + \hat{v}_k(\sqrt{1+|v|^2}+v\cdot\omega)^{-3}\right].$$

Hence

$$\int_{|\omega|=1} I\ d\omega = -\frac{1}{2}\frac{\partial^2}{\partial v_i\partial v_k}(4\pi) - \frac{\partial}{\partial v_i}(4\pi v_k) = -4\pi\delta_{ik}. \tag{5.21}$$

It follows that

$$\int_{|\omega|=1}(I+III)\ d\omega = 0. \tag{5.22}$$

Finally,

$$II = \frac{-3(\omega_i + \hat{v}_i)v_k}{[\sqrt{1+|v|^2} + v \cdot \omega]^3} = \frac{3}{2} v_k \frac{\partial}{\partial v_i} [\sqrt{1+|v|^2} + v \cdot \omega]^{-2} \tag{5.23}$$

and thus

$$\int_{|\omega|=1} II \, d\omega = 0. \tag{5.24}$$

Therefore

$$\int_{|\omega|=1} a(\omega, \hat{v}) \, d\omega = 0. \tag{5.25}$$

The computation for B is similar (cf. [9]). The most singular term B_{TT} also has a kernel with zero average. This completes the proof.

5.5. Estimates on the Particle Density

We take $f(0,x,v) = f_0(x,v) \in C_0^1$, supp $f_0 \subset \{|x| \leq k, |v| \leq k\}$. The characteristic ordinary differential equations are

$$\dot{x} = \hat{v}, \qquad \dot{v} = K, \qquad \dot{f} = 0.$$

Hence

$$f(t,x,v) = f_0(X(0,t,x,v), V(0,t,x,v))$$

and so $0 \leq f \leq \max f_0$ provided $0 \leq f_0 \leq \max f_0 < \infty$.

Next, we claim that $f(t,x,v) = 0$ if $|x| > t + k$. Indeed,

$$|X(0,t,x,v) - x| = \left| \int_0^t \hat{V}(s,t,x,v) \, ds \right| \leq t$$

and hence $|x| > k + t$ implies

$$\begin{aligned} |X(0,t,x,v)| &\geq |x| - |X(0,t,x,v) - x| \\ &> k + t - t = k. \end{aligned} \tag{5.26}$$

Thus

$$\rho(t,x) = j(t,x) = 0 \quad \text{for} \quad |x| > t + k. \tag{5.27}$$

Now we turn to estimates on derivatives of the particle density. Let $Df = \frac{\partial f}{\partial x_j}$ for any j. Then

$$(\partial_t + \hat{v} \cdot \nabla_x + K \cdot \nabla_v)(Df) = -DK \cdot \nabla_v f, \tag{5.28}$$

so that

$$\frac{d}{ds}Df(s,X(s,t,x,v),V(s,t,x,v)) = -DK\cdot\nabla_v f(s,X(s,t,x,v),V(s,t,x,v)).$$

Therefore

$$(5.29)\ |Df(t,x,v)| \le |Df(0,X(0,t,x,v),V(0,t,x,v))| + \int_0^t \Big|DK\cdot\nabla_v f(s,X(s,t,x,v),V(s,t,x,v))\Big|\,ds.$$

Now we define norms using

$$\begin{aligned}
|E(t)|_0 &= \sup_x |E(t,x)|,\\
\|E\|_0 &= \sup_{0\le t\le T} |E(t)|_0;\\
|E(t)|_1 &= \sum_{k=1}^3 \sup_x |\partial_{x_k}E(t,x)| + \sup_x |\partial_t E(t,x)|,\\
\|E\|_1 &= \sup_{0\le t\le T} |E(t)|_1;\\
|f(t)|_0 &= \sup_{x,v} f(t,x,v),\\
|f(t)|_1 &= \sup_x\sup_v \left(|\partial_t f| + \sum_{k=1}^3 (|\partial_{x_k}f| + |\partial_{v_k}f|)\right),\\
\|f\|_k &= \sup_{0\le t\le T} |f(t)|_k \quad (k=0,1)
\end{aligned}$$

with similar expressions for B.

From (5.29) it follows that

$$(5.30)\qquad |Df(t)|_0 \le c_0 + c\int_0^t (|E(\tau)|_1 + |B(\tau)|_1)|f(\tau)|_1\,d\tau.$$

A similar bound holds for $D = \frac{\partial}{\partial v_j}$ since

$$(5.31)\qquad [\partial_t + \hat{v}\cdot\nabla_x + K\cdot\nabla_v](Df) = -\frac{\partial\hat{v}}{\partial v_j}\cdot\nabla_x f - \frac{\partial\hat{v}}{\partial v_j}\times B\cdot\nabla_v f$$

and $\frac{\partial\hat{v}}{\partial v_j}$ is bounded. Therefore

$$(5.32)\qquad |f(t)|_1 \le c_0 + c_T\int_0^t [1 + |E(\tau)|_1 + |B(\tau)|_1]|f(\tau)|_1\,d\tau$$

for $t\le T$.

5.6. Bounds on the Field

Recall that, by Theorem 5.3.1, the field E can be represented as

$$E = (\text{data term}) + E_T + E_S.$$

By hypothesis we have $|v| \le \beta(t) \le \beta_T$, say, on supp f for $0 \le t \le T$. Then

$$\left|\frac{\omega_i + \hat{v}_i}{(1+|v|^2)(1+\hat{v}\cdot\omega)^2}\right| \le c_T$$

since

$$|\hat{v}\cdot\omega| \le \frac{\beta_T}{\sqrt{1+\beta_T^2}} < 1.$$

Hence with $r = |y-x|$

$$\begin{aligned}(5.33)\qquad |E_T(t,x)| &\le c_T \int_{|y-x|\le t}\int_{|v|\le\beta_T} f(t-|x-y|,y,v)\,dv\frac{dy}{r^2}\\ &\le c_T\beta_T^3 t\|f\|_0.\end{aligned}$$

For E_S we use $Sf = -K\cdot\nabla_v f = -\nabla_v\cdot(Kf)$ and integrate by parts in v:

$$\begin{aligned}(5.34)\qquad E_S &= -\iint \frac{(\omega_i+\hat{v}_i)}{1+\hat{v}\cdot\omega}(Sf)\,dv\frac{dy}{r}\\ &= -\int_{|y-x|\le t}\int \nabla_v\left[\frac{\omega_i+\hat{v}_i}{1+\hat{v}\cdot\omega}\right]\cdot(E+\hat{v}\times B)f\,dv\frac{dy}{r}.\end{aligned}$$

The v-gradient factor is bounded by the support hypothesis (by c_T, say). Therefore

$$(5.35)\qquad |E_S(t,x)| \le c_T \int_{|y-x|\le t}\int_{|v|\le\beta_T} (|E(t)|_0+|B(t)|_0)|f(t)|_0\,dv\frac{dy}{r},$$

so that

$$(5.36)\qquad |E(t)|_0 \le c_T + c_T\int_0^t (|E(\tau)|_0+|B(\tau)|_0)\,d\tau.$$

A similar estimate holds for B. Therefore by the Gronwall inequality we obtain

$$(5.37)\qquad |E(t)|_0 + |B(t)|_0 \le c_T \qquad \text{on} \quad [0,T].$$

5.7. Bounds on the Gradient of the Field

THEOREM 5.7.1 *Let* $\log^* s = \begin{cases} s & s\le 1\\ 1+\ln s & s\ge 1.\end{cases}$
Then

$$(5.38)\qquad |E(t)|_1 + |B(t)|_1 \le c_T\left\{1+\log^*\left(\sup_{\tau\le t}|f(\tau)|_1\right)\right\}$$

for $t \le T$.

Proof. Write

$$(5.39)\qquad \partial_k E^i = \binom{\text{data}}{\text{term}} + \partial_k E^i_{TT} - \partial_k E^i_{TS} + \partial_k E^i_{ST} - \partial_k E^i_{SS} + 0(1)$$

where the various terms are given in Theorem 5.4.1.

The most singular term is

$$\partial_k E^i_{TT} \equiv \int_0^t \frac{1}{t-\tau} \iint a(\omega,\hat{v}) f(\tau, x+\omega(t-\tau), v)\, dv\, d\omega\, d\tau. \tag{5.40}$$

Now $a(\omega,\hat{v})$ is bounded for $|v| \le \beta_T$. Thus for any d, $0<d<t$,

$$\int_0^{t-d} (\cdots) \le c_T \|f\|_0 \int_0^{t-d} \frac{d\tau}{t-\tau} \le c_T \ln\left(\frac{t}{d}\right) \tag{5.41}$$

and

(5.42)

$$\int_{t-d}^t (\cdots) = \int_{t-d}^t \iint a(\omega,\hat{v}) \left[f(\tau, x+\omega(t-\tau), v) - f(\tau,x,v)\right]\, d\omega\, dv \frac{d\tau}{t-\tau}$$

because $\int_{|\omega|=1} a(\omega,\hat{v})\, d\omega = 0$. Therefore

$$\left|\int_{t-d}^t (\cdots)\right| \le c\|\nabla_x f\|_0 \int_{t-d}^t \int_{|\omega|=1} \int_{|v|\le\beta_T} dv\, d\omega\, d\tau \le c_T d \|\nabla_x f\|_0. \tag{5.43}$$

Hence

$$\left|\partial_k E^i_{TT}(t,x)\right| \le c_T \left\{\ln\left(\frac{t}{d}\right) + d\|\nabla_x f\|_0\right\}. \tag{5.44}$$

Take $d^{-1} = \|\nabla_x f\|_0$ to get

$$|\partial_k E^i_{TT}| \le c_T \left\{1 + \log^*(\|\nabla_x f\|_0)\right\}. \tag{5.45}$$

(Here we are assuming that $d \equiv \|\nabla_x f\|_0^{-1} < t$; if not, a simple argument yields (5.45) in this case as well.)

For the Sf–term, we integrate by parts in v:

$$\begin{aligned} \int_{r\le t} \int b(\omega,\hat{v}) Sf\, dv \frac{dy}{r^2} &= -\int_{r\le t} \int b(\omega,\hat{v}) \nabla_v \cdot (Kf)\, dv \frac{dy}{r^2} \\ &= \int_{r\le t} \int \nabla_v b(\omega,\hat{v}) \cdot Kf\, dv\, r^{-2}\, dy. \end{aligned} \tag{5.46}$$

By the bounds already known, this term is dominated by c_T.

For the $S^2 f$–term, we write

$$\begin{aligned} S^2 f = -S[\nabla_v \cdot (Kf)] &= -(\partial_t + \hat{v}\cdot\nabla_x)\partial_{v_\ell}(fK^\ell) \\ &= -\partial_{v_\ell}[(\partial_t + \hat{v}\cdot\nabla_x)(fK^\ell)] + (\partial_{v_\ell}\hat{v}_j)\partial_{x_j}(fK^\ell) \\ &= -\nabla_v \cdot [S(Kf)] + \frac{\delta_{j\ell} - \hat{v}_j\cdot\hat{v}_\ell}{\sqrt{1+|v|^2}} (f\partial_{x_j}K^\ell + K^\ell \partial_{x_j} f). \end{aligned} \tag{5.47}$$

Recall that $c(\omega,\hat{v}) = -\frac{\omega_k(\omega_i+\hat{v}_i)}{(1+\hat{v}\cdot\omega)^2}$. Thus

$$\begin{aligned}(5.48)\quad \partial_k E^i_{SS} &\equiv \int_{r\le t}\int c(\omega,\hat{v})S^2 f\,dv\frac{dy}{r}\\ &= \int_{r\le t}\int \nabla_v c(\omega,\hat{v})\cdot S(Kf)\,dv\frac{dy}{r}\\ &\quad + \int_{r\le t}\int \underbrace{\frac{c(\omega,\hat{v})(\delta_{j\ell}-\hat{v}_j\hat{v}_\ell)}{\sqrt{1+|v|^2}}}_{\hat{c}_{j\ell}(\omega,\hat{v})}(f\partial_{x_j}K^\ell + K^\ell\partial_{x_j}f)\,dv\frac{dy}{r}.\end{aligned}$$

Now

$$\begin{aligned}(5.49)\qquad S(Kf) &= KSf + fSK\\ &= -K\nabla_v\cdot(Kf) + fSK.\end{aligned}$$

Therefore

$$\begin{aligned}(5.50)\quad \partial_k E^i_{SS} &= \iint fK\cdot\nabla_v[\nabla_v c\cdot K]\frac{dv\,dy}{r} + \iint \nabla_v c\cdot fSK\frac{dv\,dy}{r}\\ &\quad + \iint \hat{c}_{j\ell} f\partial_{x_j}K^\ell\frac{dv\,dy}{r} + \iint \hat{c}_{j\ell}K^\ell\partial_{x_j}f\frac{dv\,dy}{r}\\ &\equiv I + II + III + IV \qquad \text{respectively.}\end{aligned}$$

Now

$$\begin{aligned}(5.51)\qquad |I| &\le c_T\int_0^t (|E(\tau)|_0 + |B(\tau)|_0)^2|f(\tau)|_0\,d\tau \le c_T;\\ |II| &\le c_T\int_0^t |f(\tau)|_0(|E(\tau)|_1 + |B(\tau)|_1)\,d\tau.\end{aligned}$$

III satisfies the same bound as II. For IV, we split $\partial_{x_j}f$ again:

$$\begin{aligned}IV &= \int_{r\le t}\int \hat{c}_{j\ell}K^\ell\left[\frac{-\omega_j\nabla_v\cdot(Kf)}{1+\hat{v}\cdot\omega} + \left(\delta_{jp} - \frac{\omega_j\hat{v}_p}{1+\hat{v}\cdot\omega}\right)T_p f\right]\frac{dv\,dy}{r}\\ &\equiv IV' + IV''.\end{aligned}$$

In IV', we integrate by parts in v:

$$(5.52)\qquad |IV'| \le c_T\int_0^t (|E(\tau)|_0 + |B(\tau)|_0)^2|f(\tau)|_0\,d\tau \le c_T.$$

Also, since $|\nabla_y\omega| = 0(r^{-1})$,

$$(5.53)\qquad |IV''| \le c_T\int_0^t (|E(\tau)|_0 + |B(\tau)|_0 + |E(\tau)|_1 + |B(\tau)|_1)|f(\tau)|_0\,d\tau.$$

Combining these expressions, we get

$$(5.54)\quad |\partial_k E^i_{SS}(t)|_0 \le c_T \int_0^t \left\{ (|E(\tau)|_0 + |B(\tau)|_0)^2 + \sum_{p=0}^{1} (|E(\tau)|_p + |B(\tau)|_p) \right\} |f(\tau)|_0 \, d\tau.$$

Now by repeating the same estimates for the B field, we obtain

$$(5.55)\qquad |K(t)|_1 \le c_T \left\{ 1 + \log^* \left(\sup_{0\le\tau\le t} |f(\tau)|_1 \right) + \int_0^t |K(\tau)|_1 \, d\tau \right\}.$$

By an application of the Gronwall lemma, we deduce the Theorem.

Putting all of these estimates together, we can write

$$(5.56)\qquad \begin{aligned} |f(t)|_1 &\le c + c_T \int_0^t [1 + |K(\tau)|_1] |f(\tau)|_1 \, d\tau \\ &\le c + c_T \int_0^t \left[1 + \log^* \left(\sup_{0\le s\le\tau} |f(s)|_1 \right) \right] |f(\tau)|_1 \, d\tau. \end{aligned}$$

Put $\alpha(t) = \sup_{0\le s\le t} |f(s)|_1$; then

$$(5.57)\qquad \alpha(t) \le c + c_T \int_0^t [1 + \log^* \alpha(\tau)] \alpha(\tau) \, d\tau \equiv \zeta(t).$$

Therefore

$$(5.58)\qquad \int_c^{\zeta(t)} \frac{d\zeta}{\zeta(1 + \log^* \zeta)} \le ct.$$

This implies that $|f(t)|_1$ is bounded, and hence $|K(t)|_1$ is also bounded. This concludes the a priori estimations; when these techniques are applied to the iterates, as will be done next, Theorem 5.2.1 results.

5.8. Proof of Existence

For simplicity let us take smooth initial data $f_0(x,v)$ in C_0^2, $E_0(x)$ and $B_0(x)$ in C^3 and $E_1(x)$ and $B_1(x)$ in C^2. We recursively define the solutions $f^{(n)}(t,x,v)$, $E^{(n)}(t,x)$, $B^{(n)}(t,x)$ as follows. First, we will define $f^{(0)}(t,x,v) = f_0(x,v)$, $E^{(0)}(t,x) = E_0(x)$, $B^{(0)}(t,x) = B_0(x)$. Given the $(n-1)^{\text{st}}$ iteration, we define $f^{(n)}$ as the solution of

$$(5.59)\qquad \begin{aligned} &\partial_t f^{(n)} + \hat{v} \cdot \nabla_x f^{(n)} + \{E^{(n-1)} + \hat{v} \times B^{(n-1)}\} \cdot \nabla_v f^{(n)} = 0 \\ &f^{(n)}(0,x,v) = f_0(x,v). \end{aligned}$$

This is a *linear* equation for a single unknown (with C^2 coefficients which, as we show below, are uniformly bounded in C^1 on $0 \le t \le T$) of the form

$$(5.60)\qquad \partial_t f + a(t,x,v) \cdot \nabla_{(x,v)} f = 0$$

with initial condition f_0 where a and f_0 are C^2 functions. Thus $f^{(n)}$ is a C^2 function provided $E^{(n-1)}$ and $B^{(n-1)}$ are C^2. Now $f^{(n)}$ is constant along the characteristics of (5.59), the trajectories of the system

$$\dot{x} = \hat{v}, \quad \dot{v} = E^{(n-1)} + \hat{v} \times B^{(n-1)}. \tag{5.61}$$

Therefore $f^{(n)}$ has compact support in v, so that

$$\rho^{(n)} = \int f^{(n)} dv \quad \text{and} \quad j^{(n)} = \int \hat{v} f^{(n)} dv \tag{5.62}$$

are well-defined as C^2 functions. Then, given $f^{(n)}$ and hence $\rho^{(n)}$ and $j^{(n)}$, we define $E^{(n)}$ and $B^{(n)}$ as the solutions of the system

$$\begin{aligned} (\partial_t^2 - \Delta)E^{(n)} &= -\nabla_x \rho^{(n)} - \partial_t j^{(n)} \\ (\partial_t^2 - \Delta)B^{(n)} &= \nabla_x \times j^{(n)} \end{aligned} \tag{5.63}$$

with initial data $E_0(x),\ E_1(x),\ B_0(x),\ B_1(x)$.

LEMMA 5.8.1 *If $f^{(n)}$ is a C^2 solution of* (5.59), *and $E^{(n)}$, $B^{(n)}$ are the solutions of* (5.63) *then $E^{(n)}$ and $B^{(n)}$ are also C^2 functions.*

Proof. Since the right-hand sides of (5.63) are C^1, the solutions are C^1. To show they are C^2 we proceed by induction on n. The representation Theorem 5.3.1 can be employed to give

$$4\pi E^{(n)}(t,x) = E_0(t,x) + E_T^{(n)}(t,x) + E_S^{(n)}(t,x)$$

where $E_0(t,x)$ is the solution of the homogeneous wave equation with the same Cauchy data,

$$E_T^{(n)}(t,x) = -\iint\limits_{|y-x|\le t} \frac{(\omega+\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2} f^{(n)}(t-|y-x|,y,v)\, dv \frac{dy}{|y-x|^2}$$

and

$$E_S^{(n)}(t,x) = -\iint\limits_{|y-x|\le t} \frac{(\omega+\hat{v})}{1+\hat{v}\cdot\omega} (Sf^{(n)})(t-|y-x|,y,v)\, dv \frac{dy}{|y-x|}.$$

Now $E_0(t,x)$ is C^2. In the second integral appears the expression

$$Sf^{(n)} = -\nabla_v \cdot \{(E^{(n-1)} + \hat{v} \times B^{(n-1)}) f^{(n)}\}$$

so that we can integrate by parts with respect to v. By the induction hypothesis, $E^{(n-1)}$ and $B^{(n-1)}$ are C^2. Therefore $E^{(n)}$ is C^2, and the same holds for $B^{(n)}$. This proves Lemma 5.8.1.

We claim that the estimates (5.32) and (5.36) are valid for $f^{(n)}$, $E^{(n)}$ and $B^{(n)}$, uniformly in n. To prove this we simply repeat all the previous estimates with superscripts (n) or $(n-1)$. Thus it is clear that

$$|f^{(n)}(\tau)|_0 \leq c \tag{5.64}$$

and, as in (5.32),

$$|f^{(n)}(t)|_1 \leq c + c_T \int_0^t (1 + |E^{(n-1)}(\tau)|_1 + |B^{(n-1)}(\tau)|_1)|f^{(n)}(\tau)|_1 \, d\tau. \tag{5.65}$$

The analogue of (5.36) is

$$|E^{(n)}(t)|_0 + |B^{(n)}(t)|_0 \leq c + c \int_0^t (|E^{(n-1)}(\tau)|_0 + |B^{(n-1)}(\tau)|_0) \, d\tau, \tag{5.66}$$

for $0 \leq t \leq T$ with constants c depending on T. Iterating (5.66) we get

$$|E^{(n)}(t)|_0 + |B^{(n)}(t)|_0 \leq c(1 + ct + \cdots + c^n t^n/n!) \leq ce^{ct}. \tag{5.67}$$

Thus the fields $E^{(n)}$, $B^{(n)}$ and the density $f^{(n)}$ are pointwise bounded uniformly in n. Now the Gronwall lemma applied to (5.65) gives

$$|f^{(n)}(t)|_1 \leq c_T \exp\left[\int_0^t c(|E^{(n-1)}(\tau)|_1 + |B^{(n-1)}(\tau)|_1) \, d\tau\right]. \tag{5.68}$$

The analogue of (5.55) is

$$\begin{aligned} |E^{(n)}(t)|_1 + |B^{(n)}(t)|_1 \leq c_T &+ c_T \log^*\left(\sup_{[0,t]} |f^{(n)}(\tau)|_1\right) \\ &+ c_T \int_0^t (|E^{(n-1)}(\tau)|_1 + |B^{(n-1)}(\tau)|_1) \, d\tau. \end{aligned} \tag{5.69}$$

Putting (5.68) into (5.69), we deduce

$$|E^{(n)}(t)|_1 + |B^{(n)}(t)|_1 \leq c_T + c_T \int_0^t (|E^{(n-1)}(\tau)|_1 + |B^{(n-1)}(\tau)|_1) \, d\tau$$

since $\log^* s \leq \max\{1, 1 + \ln s\}$. This estimate is iterated as above to give a uniform bound on $\|E^{(n)}\|_1 + \|B^{(n)}\|_1$. From (5.68) follows a uniform bound for $\|f^{(n)}\|_1$, for all n and for $0 \leq t \leq T$.

With these estimates, together with compactness, it is easy to pass to the limit. But to get optimal results, it is more convenient to show directly that the sequences are Cauchy sequences in the C^1 norm. We fix two indices m and n. For $j = 0$, 1 let

$$\begin{aligned} b_j^{mn}(t) &= |E^{(m)}(t) - E^{(n)}(t)|_j + |B^{(m)}(t) - B^{(n)}(t)|_j, \\ f_j^{mn}(t) &= |f^{(m)}(t) - f^{(n)}(t)|_j. \end{aligned}$$

Exactly as in the derivation of (5.11) we have

$$b_0^{mn}(t) \le c \int_0^t \left(b_0^{m-1,n-1}(\tau) + f_0^{mn}(\tau) \right) d\tau. \tag{5.70}$$

Exactly as in (5.12) we have

$$f_0^{mn}(t) \le c \int_0^t b_0^{m-1,n-1}(\tau)\, d\tau \tag{5.71}$$

using the bounds already known, c depending on T and $0 \le t \le T$. We substitute (5.71) into (5.70) and interchange the order of integration to obtain

$$b_0^{mn}(t) \le c \int_0^t b_0^{m-1,n-1}(\tau)\, d\tau \tag{5.72}$$

with a different constant c. Iteration of (5.72) yields

$$\begin{aligned} b_0^{mn}(t) &\le c^2 \int_0^t (t-\tau) b_0^{m-2,n-2}(\tau)\, d\tau \\ &\le \cdots \\ &\le c^k \int_0^t \frac{(t-\tau)^{k-1}}{(k-1)!} b_0^{m-k,n-k}(\tau)\, d\tau \\ &\le ac^k t^k / k! \quad \text{for } m \ge k,\ n \ge k, \end{aligned}$$

where

$$b_0^{mn}(t) \le |E^{(m)}(t)|_0 + |E^{(n)}(t)|_0 + |B^{(m)}(t)|_0 + |B^{(n)}(t)|_0 \le a.$$

Therefore $E^{(n)}$, $B^{(n)}$ and, from (5.71), $f^{(n)}$ are Cauchy sequences in the C^0 norm so that they converge uniformly.

We claim that the same is valid in the C^1 norm. Let ∂ denote any first derivative of E and consider any component i. We split $\partial E^{(n)}$ and $\partial E^{(m)}$ as in the process given in Theorem 5.7.1, and then subtract these expressions. First, the TT term is written as in (5.40) – (5.42) and estimated as

$$\left| (\partial E_{TT}^{(n)} - \partial E_{TT}^{(m)})(t) \right|_0 \le c \int_0^t \left| f^{(n)}(\tau) - f^{(m)}(\tau) \right|_1 d\tau.$$

The TS and ST terms are written as in (5.46) and estimated by

$$c \int_0^t \left(b_0^{m-1,n-1}(\tau) + f_0^{mn}(\tau) \right) d\tau.$$

We break up the SS term into several pieces as in (5.50). Following the same procedures and using the known bounds in C^1, we deduce

$$\left| (\partial E_{SS}^{(n)} - \partial E_{SS}^{(m)})(t) \right|_0 \le c \int_0^t \left(b_1^{m-1,n-1}(\tau) + f_0^{mn}(\tau) \right) d\tau.$$

Thus

$$b_1^{mn}(t) \leq c \int_0^t \left(b_1^{m-1,n-1}(\tau) + f_1^{mn}(\tau) \right) d\tau. \tag{5.73}$$

Now we estimate $f_1^{mn}(t)$. For this purpose, recall the characteristic equations

$$\dot{x}_n = \hat{v}_n, \ \dot{v}_n = K^{(n-1)}(s, x_n) \tag{5.61}$$

where $K^{(n-1)} \equiv E^{(n-1)} + \hat{v}_n \times B^{(n-1)}$ is evaluated at time s. Denote the solutions of (5.61) which assume the initial values x, v when $s = t$ by $x_n(s), v_n(s)$, respectively. Below we suppress the dependence of $K^{(n)}$ on s. From the first equation in (5.61) we get

$$\left| \frac{d}{ds}(x_n - x_m) \right| \leq |v_n - v_m|$$

since the real function $p \mapsto p(1+p^2)^{-1/2}$ is has its derivative bounded by unity. The second equation in (5.61) gives

$$\begin{aligned} \left| \frac{d}{ds}(v_n - v_m) \right| &= |K^{(n-1)}(x_n) - K^{(m-1)}(x_m)| \\ &\leq |K^{(n-1)}(x_n) - K^{(n-1)}(x_m)| + |K^{(n-1)}(x_m) - K^{(m-1)}(x_m)| \\ &\leq c|x_n - x_m| + |K^{(n-1)}(x_m) - K^{(m-1)}(x_m)| \end{aligned}$$

since each $K^{(n)}$ has uniformly bounded C^1 norm. Moreover, we also have $|(K^{(n-1)} - K^{(m-1)})(x_m)| \leq \delta_{nm}$, say, where $\delta_{nm} \to 0$ as $n, m \to \infty$ uniformly on $[0, T]$ by the known bounds. Thus

$$|x_n - x_m| + |v_n - v_m| \leq c_T \left\{ \delta_{nm} + c \int_0^t (|v_n - v_m| + |x_n - x_m|)\, ds \right\}. \tag{5.74}$$

By Gronwall's inequality, the sequences $\{x_n(s)\}$, $\{v_n(s)\}$ converge uniformly on $0 \leq s \leq T$. The convergence is also uniform with respect to the parameters t, x, v where $0 \leq t \leq T$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$.

To estimate $f_1^{mn}(t)$, we only discuss x–derivatives ∂f for simplicity. Differentiating (5.59), we have

$$(\partial_t + \hat{v} \cdot \nabla_x + K^{(n-1)}(x_n) \cdot \nabla_v) \partial f^{(n)} = -\partial K^{(n-1)} \cdot \nabla_v f^{(n)}.$$

Integrating along characteristics, we can write

$$\partial f^{(n)}(t, x, v) = \partial f_0(x_n(0), v_n(0)) - \int_0^t [\partial K^{(n-1)}(x_n) \cdot \nabla_v f^{(n)}(s, x_n, v_n)]\, ds.$$

The same equation is written at stage m and the results are subtracted; this gives us the estimate

$$\left|(\partial f^{(n)} - \partial f^{(m)})(t,x,v)\right| \le |\partial f_0(x_n(0), v_n(0)) - \partial f_0(x_m(0), v_m(0))| + \int_0^t |\partial K^{(n-1)}(x_n)\cdot\nabla_v f^{(n)}(s,x_n,v_n) - \partial K^{(m-1)}(x_m)\cdot\nabla_v f^{(m)}(s,x_m,v_m)|\,ds.$$

By the hypothesis on f_0, the first term tends to zero as $n, m \to \infty$. Therefore we can write

$$\begin{aligned}
&\left|(\partial f^{(n)} - \partial f^{(m)})(t,x,v)\right| \\
&\le \varepsilon_{nm} + \int_0^t \Big[|\partial K^{(n-1)}(x_m)\cdot(\nabla_v f^{(n)}(s,x_n,v_n) - \nabla_v f^{(n)}(s,x_m,v_m))| \\
&\quad + |(\partial K^{(n-1)}(x_m) - \partial K^{(m-1)}(x_m))\cdot\nabla_v f^{(n)}(s,x_m,v_m)| \\
&\quad + |\partial K^{(m-1)}(x_m)\cdot(\nabla_v f^{(n)}(s,x_m,v_m) - \nabla_v f^{(m)}(s,x_m,v_m))|\Big]\,ds
\end{aligned}$$

where $\varepsilon_{nm} \to 0$ as $m, n \to \infty$, uniformly for $0 \le s \le T$, $0 \le t \le T$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$. The first term in the last integral tends to zero uniformly on $[0,T]$ as a consequence of (5.74) and the known C^1 bounds. The second term in the integrand is dominated by $cb_1^{m-1,n-1}(s)$, and the last by $cf_1^{mn}(s)$. We know that the v derivatives of the difference can be estimated in terms of the x derivatives, as in (5.31). Therefore

$$f_1^{mn}(t) \le \varepsilon'_{nm} + c\int_0^t \left[b_1^{m-1,n-1}(s) + f_1^{mn}(s)\right] ds \tag{5.75}$$

where $\varepsilon'_{nm} \to 0$ uniformly on $[0,T]$ as $m, n \to \infty$. The function

$$G_{nm}(t) \equiv \int_0^t f_1^{mn}(s)\,ds$$

then satisfies

$$\dot G_{nm}(t) - cG_{nm}(t) \le \varepsilon'_{nm} + c\int_0^t b_1^{m-1,n-1}(s)\,ds$$

so that

$$G_{nm}(t) \le \varepsilon'_{nm}\int_0^t e^{c(t-s)}ds + c\int_0^t e^{c(t-s)}\int_0^s b_1^{m-1,n-1}(\tau)\,d\tau\,ds.$$

Using this in (5.75), we get

$$f_1^{mn}(t) \le \varepsilon''_{nm} + c\int_0^t b_1^{m-1,n-1}(s)\,ds \tag{5.76}$$

with a different constant c (depending on T), and where $\varepsilon''_{nm} \to 0$ uniformly on $[0,T]$ when $n,m \to \infty$. Substituting (5.76) into (5.73), we arrive at the inequality

$$b_1^{mn}(t) \le \delta_{nm} + c\int_0^t b_1^{m-1,n-1}(s)\,ds \tag{5.77}$$

again with a different constant c (depending on T) and another expression δ_{nm} which tends to zero uniformly on $[0,T]$ when $n,m \to \infty$. We can easily iterate (5.77) to get

$$\begin{aligned} b_1^{mn}(t) \le \delta_{nm}\left(1 + ct + \frac{c^2t^2}{2} + \cdots + \frac{c^{l-1}t^{l-1}}{(l-1)!}\right) \\ + \frac{c^l}{(l-1)!}\int_0^t (t-\tau)^{l-1} b_1^{m-l,n-l}(\tau)\,d\tau. \end{aligned}$$

Calling b an upper bound for the C^1–norm of the field, we thus have

$$b_1^{m,n}(t) \le \delta_{nm}e^{cT} + \frac{bc^lT^l}{l!} \quad \text{on } 0 \le t \le T$$

for $m,n \ge l$. Therefore $E^{(n)}$, $B^{(n)}$ and, from (5.76), $f^{(n)}$ are Cauchy sequences in the C^1 norm.

Call the limits E, B and f, respectively. Then $E^{(n)} \to E$, $B^{(n)} \to B$ and $f^{(n)} \to f$ uniformly for $t \in [0,T]$, $x \in \mathbb{R}^3$, $v \in \mathbb{R}^3$, together with all their first derivatives. Passage to the limit in (5.59) yields the Vlasov equation. Passage to the limit in (5.63) yields the second-order wave equations for E, B taken in the sense of distributions with charge and current given by $\rho = \int f\,dv$, $j = \int \hat{v} f\,dv$.

This proves Theorem 5.2.1 for smooth initial data. If f_0, E_1 and B_1 are merely C^1 functions and E_0 and B_0 are merely C^2, we approximate them by smoother (say C^∞) functions and pass to the limit as in the proof just completed. The details are almost the same as what we have already done, but a little simpler. This completes the proof of Theorem 5.2.1 for the simplified case of a single species.

Of course no plasma is composed of a single species. The simplest way to study a real problem is to let f denote the density of electrons, and to introduce a known "background" ion density $n(x)$, and to write the charge density as

$$\rho = \int f\,dv - n(x).$$

We require *neutrality* which is the condition that the average value of ρ is zero, i.e., that

$$\int_{\mathbb{R}^3\times\mathbb{R}^3} f_0\,dx\,dv = \int_{\mathbb{R}^3} n(x)\,dx$$

(recall that the integral of f over $\mathbb{R}^3 \times \mathbb{R}^3$ is invariant). This change introduces some extra terms involving n into the earlier calculations, but each such term is harmless and can be easily estimated.

More accurately, we should study several species with densities f_α which satisfy (5.3). Each velocity of propagation is different

$$\hat{v}_\alpha = \frac{v}{\sqrt{m_\alpha^2 + |v|^2/c^2}}$$

and hence the operator S now depends on α:

$$S_\alpha = \partial_t + \hat{v}_\alpha \cdot \nabla_x.$$

Each f_α remains pointwise bounded. In the representations for the fields and their derivatives, the source terms are replaced by

$$\rho = 4\pi \int \sum_\alpha e_\alpha f_\alpha \, dv, \qquad j = 4\pi \int \sum_\alpha \hat{v}_\alpha e_\alpha f_\alpha \, dv.$$

(Recall that e_α is the charge of particles of species α). Finally the differences $f_\alpha^n - f_\alpha^m$ are estimated for each α separately. These are the only modifications necessary to complete the proof for several species.

In [11] the following generalization is established. The hypotheses of Theorem 5.2.1 require compact support in v. The result remains valid under the following condition: on every interval $[0, T]$,

$$\sup_{\alpha, x, t \leq T} \int \sqrt{1 + |v|^2} \, f_\alpha(t, x, v) \, dv < \infty.$$

Thus some particles may have unbounded momentum.

The classical (non-relativistic) Vlasov-Maxwell system may also be treated by the same method. Notice that in the decomposition of derivatives leading to the representation of the field (Theorem 5.3.1), it was essential that expressions such as $1 + \hat{v} \cdot \omega$ could be bounded away from 0. In the nonrelativistic problem, the corresponding expression is $1 + v \cdot \omega$. Thus singularities may appear on a larger set of v's, and at *finite* momenta as well. Hence smooth global existence in the non-relativistic case seems problematic, at least by this approach.

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Chapter 6

DILUTE COLLISIONLESS PLASMAS

6.1. The Small–data Theorem

In this chapter we consider small–data solutions to the Cauchy problem for the Relativistic Vlasov–Maxwell system (cf. [38]). Assume that we have several species with masses m_α and charges e_α, $1 \leq \alpha \leq N$. The *relativistic velocity* is

$$\hat{v}_\alpha = \frac{v}{\sqrt{m_\alpha^2 + |v|^2/c^2}}$$

where c is the speed of light. The particle densities $f_\alpha(t,x,v)$ and fields E, B satisfy

$$\partial_t f_\alpha + \hat{v}_\alpha \cdot \nabla_x f_\alpha + e_\alpha \left(E + \frac{\hat{v}_\alpha}{c} \times B \right) \cdot \nabla_v f_\alpha = 0, \tag{6.1}$$

$$\rho = 4\pi \int \sum_\alpha e_\alpha f_\alpha \, dv, \qquad j = 4\pi \int \sum_\alpha \hat{v}_\alpha e_\alpha f_\alpha \, dv,$$

$$\begin{aligned} \partial_t E &= c\nabla \times B - j \qquad & \nabla \cdot E &= \rho \\ \partial_t B &= -c\nabla \times E \qquad & \nabla \cdot B &= 0. \end{aligned}$$

Given are $f_\alpha(0,x,v) = f_{\alpha 0}(x,v)$, $E(0,x) = E_0(x)$, $B(0,x) = B_0(x)$ satisfying $\nabla \cdot E_0 = \rho_0$, $\nabla \cdot B_0 = 0$, $\int \rho_0 \, dx = 0$.

In Chapter 5 a sufficient condition was given under which smooth global solutions are known to exist. Namely, given any $T > 0$, a bound on the v–support of the particle densities $f_\alpha(t,x,v)$ for all x, α, $t \leq T$ is sufficient. We show here that if the initial data of the fields and of the $\{f_\alpha\}$ are small, such a bound may be achieved. This material is essentially the content of [18].

THEOREM 6.1.1 *For each $k > 0$, there exist constants $\varepsilon_0 > 0$ and $\beta > 0$ with the following property. Let $f_{\alpha 0}(x,v)$ ($\alpha = 1, 2, \ldots, N$) be non–negative C^1 functions with supports in $\{|x| \leq k\}$, $\{|v| \leq k\}$. Let $E_0(x)$, $B_0(x)$ be C^2*

functions with supports in $\{|x| \leq k\}$ *which satisfy the constraints*

$$\nabla \cdot E_0 = \rho_0 \equiv 4\pi \int_{\mathbb{R}^3} \sum_\alpha e_\alpha f_{\alpha 0}\, dv, \quad \nabla \cdot B_0 = 0, \quad \int_{\mathbb{R}^3} \rho_0\, dx = 0. \tag{6.2}$$

If the data satisfy

$$\sum_\alpha \|f_{\alpha 0}\|_{C^1} + \|E_0\|_{C^2} + \|B_0\|_{C^2} \leq \varepsilon_0, \tag{6.3}$$

then there exists a unique solution of (6.1) *for all* $x \in \mathbb{R}^3$ *and all* $v \in \mathbb{R}^3$ *and all times* t, $0 \leq t < \infty$, *with* $f_\alpha, E, B \in C^1$ *having initial data* $f_{\alpha 0}, E_0, B_0$ *such that*

$$f_\alpha(t,x,v) = 0 \quad \textit{for} \quad |v| \geq \beta \quad \textit{for all } \alpha, t \textit{ and } x. \tag{6.4}$$

Furthermore, $E(t,x) = B(t,x) = 0$ *for* $|x| > ct + k$. *For all* $\varepsilon > 0$, *there exists* $\varepsilon_0 > 0$ *such that if (6.3) holds, then*

$$|E(t,x)| + |B(t,x)| \leq \frac{\varepsilon}{(t+1)(ct - |x| + 2k)} \tag{6.5}$$

for all $t \geq 0, x \in \mathbb{R}^3$.

The key step in the proof is to show that the paths of the particles spread out with time. Hence only a small set of momenta (of diameter $O(1/t)$ in momentum space as $t \to \infty$) could reach a given point (t,x) from the support of the initial data. Then $\rho(t,x)$, which is an integral over a set of momenta of volume $O(t^{-3})$, is itself of that order. Since the particle paths are given by the equations

$$\dot{x} = \hat{v}_\alpha, \quad \dot{v} = e_\alpha \left(E + c^{-1}\hat{v}_\alpha \times B\right), \tag{6.6}$$

the particles would move in approximately straight lines if E and B were small. Thus we must also prove that the electromagnetic field decays as $t \to \infty$.

In the case of the Vlasov–Poisson system there is no explicit time dependence in the field equation ($B \equiv 0$, $\Delta u = \rho$, $E = \nabla u$) so that Bardos and Degond [1] were able to iterate in a space where E decays in L^∞ at the rate $O(t^{-2})$ and ∇E at the rate $O(t^{-5/2})$. For (6.1), the field satisfies inhomogeneous wave equations like

$$(\partial_t^2 - \Delta)E = -\nabla_x \rho - \partial_t j. \tag{6.7}$$

So the best possible L^∞ rate of decay for the field (with general functions ρ, j) is $O(t^{-1})$, which is far too slow for the methods of [1] to succeed. Horst [23], on the other hand, imposed conditions as $t \to \infty$ designed to allow more rapid decay of the field. For the present problem with arbitrary initial conditions and a magnetic field, we introduce a weighted L^∞ norm for the field, as was pioneered by John [25]. We use the weight

$$(ct + |x| + 2k)(ct - |x| + 2k).$$

By causality we know that $|x| \le ct + k$ on the support of the field, so that the second factor simply introduces an extra decay factor of t^{-1} inside the light cone. In order to exploit this extra factor when solving (6.1), one must show that $|x|$ stays well inside the cone. Since $\dot{x} = \hat{v}_\alpha$ this is true if the momenta v remain bounded. This means that the Vlasov and Maxwell characteristics are well separated; that is, the system (6.1) is effectively strictly hyperbolic. Thus we have come full circle to a condition on the behavior of the momenta. To carry out the process we also need to estimate the spatial derivatives of the field, for which a similar weight function is used.

6.2. Outline of the proof

The main structure of the proof, as exposed in the last chapter, follows [15]. In particular, uniqueness was established, and for existence the following construction was utilized. For given functions $E^{(0)}(t,x)$ and $B^{(0)}(t,x)$, we define $E^{(n)}(t,x), B^{(n)}(t,x)$ and $f_\alpha^{(n)}(t,x,v)$ inductively as follows. Given the $(n-1)^{st}$ iteration, we define $f_\alpha^{(n)}$ as the solution of the linear equation

$$\partial_t f_\alpha^{(n)} + \hat{v}_\alpha \cdot \nabla_x f_\alpha^{(n)} + e_\alpha \left(E^{(n-1)} + \hat{v}_\alpha \times B^{(n-1)}\right) \cdot \nabla_v f_\alpha^{(n)} = 0 \tag{6.8}$$

$$f_\alpha^{(n)}(0,x,v) = f_{\alpha 0}(x,v).$$

For simplicity we have set the speed of light equal to unity ($c = 1$). Then we define

$$\rho^{(n)} = 4\pi \sum_\alpha e_\alpha \int f_\alpha^{(n)}\, dv, \quad j^{(n)} = 4\pi \sum_\alpha e_\alpha \int \hat{v}_\alpha f_\alpha^{(n)}\, dv. \tag{6.9}$$

Finally we define $E^{(n)}, B^{(n)}$ as the solution of Maxwell's equations

$$\begin{aligned} \partial_t E^{(n)} &= \nabla \times B^{(n)} - j^{(n)} \qquad & \nabla \cdot E^{(n)} &= \rho^{(n)}, \\ \partial_t B^{(n)} &= -\nabla \times E^{(n)} & \nabla \cdot B^{(n)} &= 0 \end{aligned}$$

with data $E^{(n)}(0,x) = E_0(x),\ B^{(n)}(0,x) = B_0(x)$.

A consequence of Theorem 5.2.1 from the last chapter is that if there exists $\beta > 0$, independent of t, x, α and n, such that

$$f_\alpha^{(n)}(t,x,v) = 0 \quad \text{for} \quad |v| > \beta, \tag{6.10}$$

then $(f_\alpha^{(n)}, E^{(n)}, B^{(n)})$ converge to a C^1 solution (f_α, E, B) of (6.1). The rest of this chapter is devoted to proving (6.10) under the "smallness condition" (6.3).

We abbreviate the field as $K(t,x) =$ the pair $(E(t,x), B(t,x))$. Define the norms

$$\|K\|_0 = \sup_{x,t}(t+|x|+2k)(t-|x|+2k)\{|E(t,x)|+|B(t,x)|\}, \tag{6.11}$$

$$\|K\|_1 = \sup_{x,t}\frac{(t+|x|+2k)(t-|x|+2k)^2}{\ln(t+|x|+2k)}\{|\nabla_x E(t,x)|+|\nabla_x B(t,x)|\}, \tag{6.12}$$

and $\|K\| = \|K\|_0 + \|K\|_1$. Let $\varepsilon > 0$ and let

$$\mathcal{K} \equiv \{K | K \quad \text{is} \quad C^1, \; K = 0 \quad \text{for} \quad |x| > t+k, \; \|K\| \leq \varepsilon\}.$$

Given $K \in \mathcal{K}$, we define the characteristics as the solutions $X = X_\alpha(s,t,x,v)$, $V = V_\alpha(s,t,x,v)$ of the ordinary differential system (6.6), that is,

$$\frac{\partial X}{\partial s} = \hat{V} \equiv V(m_\alpha^2 + |V|^2)^{-1/2}, \tag{6.13}$$

$$\frac{\partial V}{\partial s} = e_\alpha\left(E(s,X) + \hat{V} \times B(s,X)\right), \tag{6.14}$$

with the "initial" conditions $X_\alpha(t,t,x,v) = x$ and $V_\alpha(t,t,x,v) = v$.

Next we define

$$f_\alpha(t,x,v) = f_{\alpha 0}(X_\alpha(0,t,x,v),\ V_\alpha(0,t,x,v)). \tag{6.15}$$

Thus $f_\alpha(t,x,v)$ is the solution of the Vlasov equation,

$$\partial_t f_\alpha + \hat{v}_\alpha \cdot \nabla_x f_\alpha + e_\alpha(E + \hat{v}_\alpha \times B) \cdot \nabla_v f_\alpha = 0, \tag{6.16}$$

with the initial condition $f_\alpha(0,x,v) = f_{\alpha 0}(x,v)$. We define ρ and j as in (6.9) and define $K^* = (E^*, B^*)$ as the solution of Maxwell's equations

$$\begin{aligned} \partial_t E^* &= \nabla \times B^* - j \qquad & \nabla \cdot E^* &= \rho, \\ \partial_t B^* &= -\nabla \times E^* \qquad & \nabla \cdot B^* &= 0 \end{aligned}$$

with the initial conditions $E^*(0,x) = E_0(x)$, $B^*(0,x) = B_0(x)$.

Thus the iteration scheme may be summarized as $K^{(n)} = (K^{(n-1)})^*$. We begin the scheme by defining $K^{(0)} \equiv 0$ (that is, $E^{(0)}(t,x) \equiv B^{(0)}(t,x) \equiv 0$). We shall prove the following two theorems.

THEOREM 6.2.1 *If $K \in \mathcal{K}$ and ε is small enough, then there exists $\beta > 0$ depending only on k, ε and ε_0 such that $f_\alpha(t,x,v) = 0$ for $|v| \geq \beta$ and for all α, x, t.*

THEOREM 6.2.2 *If $K \in \mathcal{K}$ and ε is small enough, then $K^* \in \mathcal{K}$.*

Proof of Theorem 6.1.1. We define the sequences $f_\alpha^{(n)}, K^{(n)}$ as above. Since $K^{(0)} \in \mathcal{K}$, Theorem 6.2.2 states that $K^{(n)} \in \mathcal{K}$ for all n. By Theorem 6.2.1, $f_\alpha^{(n)} = 0$ for $|v| \geq \beta$. By the work from Chapter 5 ([15]), $f_\alpha^{(n)}, K^{(n)}$ and their first derivatives converge pointwise to f and K. Therefore $K \in \mathcal{K}$. Hence (6.5) is valid, as well as

$$|\nabla_x E(t,x)| + |\nabla_x B(t,x)| \leq \frac{\varepsilon \ln(2+t)}{(1+t)(t-|x|+2k)^2}$$

for all $t \geq 0, x \in \mathbb{R}^3$. As was mentioned earlier, (f, K) is a solution of (6.1).

6.3. Characteristics

The characteristics are curves defined by the solutions to (6.13) and (6.14). They exist as C^1 functions of s, t, x, v for some time $0 \leq t < T^*, 0 \leq s < T^*$ because E and B are C^1 functions. For as long as the characteristics exist, we define

$$P(t) = \sup\Big\{|V_\alpha(s,0,x,v)| : |x| \leq k,\ |v| \leq k,\ 0 \leq s \leq t,\ 1 \leq \alpha \leq N\Big\},$$

that is, the largest momentum up to time t emanating from the support of $f_{\alpha 0}$. Then $P(t)$ is a continuous function of t for $0 \leq t < T^*$.

Before estimating $P(t)$ we motivate its definition. In this brief section we will drop the dependence on the species through the parameter α. By the above definitions, we have

$$\begin{aligned} X(t,0,X(0,t,x,v),V(0,t,x,v)) &= x, \\ V(t,0,X(0,t,x,v),V(0,t,x,v)) &= v. \end{aligned} \tag{6.17}$$

Set $y = X(0,t,x,v)$, $w = V(0,t,x,v)$. Then these equations give us

$$x = X(t,0,y,w), \quad v = V(t,0,y,w).$$

Similarly, by uniqueness, one has

$$\begin{aligned} X(s,0,X(0,t,x,v),V(0,t,x,v)) &= X(s,t,x,v) \\ V(s,0,X(0,t,x,v),V(0,t,x,v)) &= V(s,t,x,v). \end{aligned} \tag{6.18}$$

Since f is constant on characteristics, we have

$$f(t,x,v) = f_0(X(0,t,x,v),V(0,t,x,v))$$

so that the support of f can be calculated via

$$\begin{aligned} &\{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 : f(t,x,v) \neq 0\} \\ &= \{(x,v) \in \mathbb{R}^3 \times \mathbb{R}^3 : f_0(X(0,t,x,v),V(0,t,x,v)) \neq 0\} \\ &= \Big\{\Big(X(t,0,X(0,t,x,v),V(0,t,x,v)), V(t,0,X(0,t,x,v),V(0,t,x,v))\Big) : \\ &\qquad f_0(X(0,t,x,v),V(0,t,x,v)) \neq 0\Big\}. \end{aligned}$$

This explains the extent of the v–support of f as well as the definition of $P(t)$.

LEMMA 6.3.1 *If* $0 \leq s \leq t$ *and* $f_\alpha(t,x,v) \neq 0$, *then*

$$s - |X_\alpha(s,t,x,v)| + 2k \geq (k+s)(2+2P^2(t))^{-1}.$$

Proof. For $0 \leq s \leq t$ assume that

$$f_\alpha(t,x,v) = f_{\alpha 0}(X_\alpha(0,t,x,v),\, V_\alpha(0,t,x,v)) \neq 0$$

and let

$$y_1 = X_\alpha(0,t,x,v), \quad w_1 = V_\alpha(0,t,x,v).$$

Then $|y_1| = |X_\alpha(0,t,x,v)| \leq k$ and $|w_1| = |V_\alpha(0,t,x,v)| \leq k$. We have from the second part of (6.18)

$$|V_\alpha(s,t,x,v)| \leq P(t). \tag{6.19}$$

Hence $|v| = |V_\alpha(t,0,y_1,w_1)| \leq P(t)$ and

$$|X_\alpha(s,t,x,v)| \leq |X_\alpha(0,t,x,v)| + \int_0^s |\widehat{V}_\alpha(\tau,t,x,v)|\, d\tau \leq k + s\widehat{P}(t),$$

where, assuming $m_\alpha = e_\alpha = 1$ for simplicity,

$$\widehat{P}(t) = P(t)(1+P^2(t))^{-1/2} < 1.$$

But

$$1 - \widehat{P}(t) = (1+P^2(t))^{-1/2}\{(1+P^2(t))^{1/2} + P\}^{-1} \geq \frac{1}{2(1+P^2(t))}.$$

So

$$|X_\alpha(s,t,x,v)| \leq k + s - s\{2(1+P^2(t))\}^{-1}$$

which implies Lemma 6.3.1.

LEMMA 6.3.2 *If* $K \in \mathcal{K}$ *and if* ε *is sufficiently small, say* $\varepsilon < \varepsilon_1$, *then the characteristics* $X_\alpha(s), V_\alpha(s)$ *exist for all* s *(*T^* *is infinite) and* $P(t)$ *is bounded, say* $P(t) \leq \beta$. *Here* ε_1 *and* β *depend only on* k. *Therefore, if* $f_\alpha(t,x,v) \neq 0$ *for some* (α,t,x), *then* $|v| \leq \beta$.

Proof. We have for $t \geq 0$, writing $X(s) = X_\alpha(s,t,x,v)$ and $V(s) = V_\alpha(s,t,x,v)$,

$$\begin{aligned}
|V_\alpha(0,t,x,v) - v| &\leq \int_0^t \left|E(s,X(s)) + \widehat{V}(s) \times B(s,X(s))\right| ds \\
&\leq \|K\|_0 \int_0^t (s+|X(s)|+2k)^{-1}(s-|X(s)|+2k)^{-1}\, ds \\
&\leq \|K\|_0 \int_0^t (s+k)^{-2}(2+2P^2(t))\, ds \\
&\leq 2k^{-1}(1+P^2(t))\|K\|_0
\end{aligned}$$

by Lemma 6.3.1, provided $|X_\alpha(0,t,x,v)| \le k$ and $|V_\alpha(0,t,x,v)| \le k$. Let $y = X_\alpha(0,t,x,v)$ and $w = V_\alpha(0,t,x,v)$, so that $x = X_\alpha(t,0,y,w)$ and $v = V_\alpha(t,0,y,w)$. Then

$$|w - V_\alpha(t,0,y,w)| \le 2k^{-1}(1 + P^2(t))\|K\|_0$$

for $|w| \le k$ and $|y| \le k$. Thus

$$P(t) \le k + 2k^{-1}(1 + P^2(t))\|K\|_0.$$

If $\|K\|_0$ is sufficiently small (depending on k), this implies that $P(t)$ is a bounded function of t. (See the argument in Chapter 2 surrounding Fig. 2.1).

Proof of Theorem 6.2.1. By Lemma 6.3.2, the characteristics exist for all time. If $f_\alpha(t,x,v) \neq 0$ for some (x,t,α,v), then, in view of (6.15), $|y| = |X_\alpha(0,t,x,v)| \le k$ and $|w| = |V_\alpha(0,t,x,v)| \le k$. By the definition of $P(t)$,

$$|v| = |V_\alpha(t,0,y,w)| \le P(t) \le \beta.$$

LEMMA 6.3.3 *If ε is sufficiently small, there is a constant $c > 0$ such that*

$$|X_\alpha(0,t,x,v) - X_\alpha(0,t,x,w)| \ge ct|v - w|$$

for all t, x, v and w such that $f_\alpha(t,x,v) \neq 0$ and $f_\alpha(t,x,w) \neq 0$.

Proof. For simplicity, since α is fixed, we drop the subscript α. Following Horst [23], we rewrite the characteristic equation for dv/dt in terms of $\hat{v}$, as follows:

$$\text{(6.20)} \qquad \frac{d\hat{v}}{dt} = m^{-1}(1 - |\hat{v}|^2)^{1/2}\{E + \hat{v} \times B - (\hat{v} \cdot E)\hat{v}\} \equiv J(t,x,\hat{v}).$$

Since $(1-|\hat{v}|^2)^{1/2} = m(m^2+|v|^2)^{-1/2} < 1$, the derivatives of J can be estimated as

$$\begin{aligned} |\nabla_x J(t,x,\hat{v})| &\le c(1 - |\hat{v}|^2)^{1/2}|\nabla_x K| \\ &\le c\|K\|_1(t + r + 2k)^{-1}\ln(t + r + 2k)(t - r + 2k)^{-2}, \end{aligned}$$

where $r = |x|$, and

$$\begin{aligned} |\nabla_{\hat{v}} J(t,x,\hat{v})| &\le c(1 - |\hat{v}|^2)^{-1/2}|K| \le c(1 + |v|^2)^{1/2}|K| \\ &\le c(1 + |v|^2)^{1/2}\|K\|_0(t + r + 2k)^{-1}(t - r + 2k)^{-1}. \end{aligned}$$

For the characteristic passing through the point (t,x,v), where $f(t,x,v) \neq 0$, we abbreviate $X(s) = X(s,t,x,v)$ and $V(s) = V(s,t,x,v)$. Then we substitute s for $t, X(s)$ for x and $V(s)$ for v to obtain

$$\text{(6.21)} \qquad \left|(\nabla_x J)(s, X(s), \widehat{V}(s))\right| \le c(1 + \beta^2)^2\|K\|(k + s)^{-3}\ln(k + s) \equiv g(s),$$

and

$$(6.22)\qquad \left|(\nabla_{\hat{v}}J)(s,X(s),\widehat{V}(s))\right| \le c(1+\beta^2)^{3/2}\|K\|(k+s)^{-2} \equiv h(s)$$

by Lemma 6.3.1.

Now let t, x, v and w be fixed so that $f(t,x,v) \neq 0$ and $f(t,x,w) \neq 0$. For brevity we denote $X_1(s) = X(s,t,x,v)$ and $X_2(s) = X(s,t,x,w)$. By (6.13) and (6.20) we have, for $0 \le \xi \le t$,

$$\begin{aligned}
X_1(\xi) - X_2(\xi) + (t-\xi)(\hat{v}-\hat{w}) &= \int_\xi^t [-\widehat{V}_1(\tau) + \hat{v} + \widehat{V}_2(\tau) - \hat{w}]\, d\tau \\
&= \int_\xi^t \int_\tau^t \left[\frac{\partial \widehat{V}_1}{\partial s} - \frac{\partial \widehat{V}_2}{\partial s}\right] ds\, d\tau \\
&= \int_\xi^t \int_\tau^t \left[J(s,X_1(s),\widehat{V}_1(s)) - J(s,X_2(s),\widehat{V}_2(s))\right] ds\, d\tau.
\end{aligned}$$

Hence by (6.21) and (6.22), we have

$$|X_1(\xi) - X_2(\xi) + (t-\xi)(\hat{v}-\hat{w})| \le D(\xi),$$

where

$$\begin{aligned}
D(\xi) &\equiv \int_\xi^t \left|-\widehat{V}_1(\tau) + \hat{v} + \widehat{V}_2(\tau) - \hat{w}\right| d\tau \\
&\le \int_\xi^t \int_\tau^t \left\{g(s)|X_1(s) - X_2(s)| + h(s)|\widehat{V}_1(s) - \widehat{V}_2(s)|\right\} ds\, d\tau \\
&\le \int_\xi^t \int_\tau^t \left\{g(s)\int_s^t |\widehat{V}_1(\theta) - \widehat{V}_2(\theta)|\, d\theta + h(s)|\widehat{V}_1(s) - \widehat{V}_2(s)|\right\} ds\, d\tau \\
&\le \int_\xi^t \left[\int_\tau^t g(s)\left\{D(s) + (t-s)|\hat{v}-\hat{w}|\right\} ds + h(\tau)D(\tau) \right. \\
&\qquad \left. + \int_\tau^t h(s)\, ds|\hat{v}-\hat{w}|\right] d\tau.
\end{aligned}$$

In the first part of this expression, we switch the order of integration, obtaining the factor $(s-\xi) \le s$. In terms of

$$I = \int_0^\infty [sg(s) + h(s)]\, ds,$$

we therefore find the estimate

$$D(\xi) \le \int_\xi^t [sg(s) + h(s)]D(s)\, ds + (t-\xi)|\hat{v}-\hat{w}|I.$$

After changing variables $\xi \to (t-\xi)$, we apply Gronwall's inequality to obtain

$$D(\xi) \le (t-\xi)|\hat{v}-\hat{w}|I \exp\left[\int_\xi^t (sg(s)+h(s))\, ds\right] \le (t-\xi)|\hat{v}-\hat{w}|I \exp I.$$

We choose ε so small that $I \exp I < 1/2$, recalling that $\|K\| < \varepsilon$. Choosing $\xi = 0$, we therefore have

$$|X_1(0) - X_2(0) + t(\hat{v} - \hat{w})| \leq D(0) \leq (t/2)|\hat{v} - \hat{w}|.$$

Therefore

$$|X_1(0) - X_2(0)| \geq (t/2)|\hat{v} - \hat{w}|. \tag{6.23}$$

Now $v = m\hat{v}(1 - |\hat{v}|^2)^{-1/2}$, so that

$$|\partial v_i/\partial \hat{v}_j| \leq c(1 - |\hat{v}|^2)^{-3/2} = c(m^2 + |v|^2)^{+3/2}$$

for each i and j. Since $f(t, x, v) \neq 0$, we have $|X_1(0)| \leq k$ and $|V_1(0)| \leq k$, whence $|v| \leq P(t) \leq \beta$ by the second part of (6.17) and Lemma 6.3.2. Similarly $|w| \leq \beta$. Therefore

$$|v - w| \leq c(1 + \beta)^3|\hat{v} - \hat{w}|.$$

By (6.23), we have $|X_1(0) - X_2(0)| \geq ct|v - w|$.

A brief remark is in order. If one attempts to remove the support assumption on f_0 in v, a technical problem appears at exactly this juncture.

6.4. The Particle Densities

We assume that ε is small enough for the validity of Lemmas 6.3.2 and 6.3.3.

LEMMA 6.4.1 *There is a positive constant c such that*

$$\int f_\alpha(t, x, v)\, dv \leq c\|f_0\|_0(1 + t)^{-3} \leq c\|f_0\|_0(t + |x| + 2k)^{-3}$$

for all $t \geq 0$, where $\|f_0\|_0 \equiv \sup_{\alpha,x,v} |f_{\alpha 0}(x, v)|$.

Proof. We recall that $f_\alpha(t, x, v) = f_{\alpha 0}(X, V)$, where $X = X_\alpha(0, t, x, v)$ and $V = V_\alpha(0, t, x, v)$. We have $|X| \leq k$ and $|V| \leq k$ so that $|v| \leq \beta$ by Lemma 6.3.2. So Lemma 6.4.1 is valid for $t \leq 1$. Now the integration in $\int f_\alpha\, dv$ may be taken over the set $\mathcal{A} = \{v : |X| \leq k\}$. By Lemma 6.3.3 the diameter of $\mathcal{A}$ is at most $2k/(ct)$, and so its volume is at most $O(t^{-3})$. Therefore $\int f_\alpha\, dv$ is at most $c\|f_0\|_0 t^{-3}$. The last inequality follows from the vanishing of $f_\alpha(t, x, v)$ for $|x| > t + k$.

LEMMA 6.4.2 *There is a positive constant c such that*

$$|\nabla_x f_\alpha(t, x, v)| \leq c\|f_0\|_1$$

and

$$|\nabla_v f_\alpha(t, x, v)| \leq c(1 + t)\|f_0\|_1,$$

where

$$\|f_0\|_1 = \sup_{\alpha,x,v} \{|f_{\alpha 0}(x, v)| + |\nabla_x f_{\alpha 0}(x, v)| + |\nabla_v f_{\alpha 0}(x, v)|\}.$$

Proof. Write $L_\alpha = \partial_t + \hat{v}_\alpha \cdot \nabla_x + (E + \hat{v}_\alpha \times B) \cdot \nabla_v$, so that the Vlasov equation is $L_\alpha f_\alpha = 0$. Again we drop the subscript α. Write $X(s) = X(s,t,x,v)$ and $V(s) = V(s,t,x,v)$ as before. Fix a coordinate x_j and let $\partial f = \partial f/\partial x_j$ for brevity. Then

$$L(\partial f) = -(\partial E + \hat{v} \times \partial B) \cdot \nabla_v f \equiv h(t,x,v).$$

Thus

$$\frac{d}{ds}[\partial f(s, X(s), V(s))] = h(s, X(s), V(s)).$$

Integrating from 0 to $\tau \le t$, we have

$$\partial f(\tau, X(\tau), V(\tau)) = \partial f_0(X(0), V(0)) + \int_0^\tau h(s, X(s), V(s))\, ds,$$

so that

$$\begin{aligned} |\nabla_x f(\tau, X(\tau), V(\tau))| &\le \|f_0\|_1 + \|K\|_1 \int_0^\tau \frac{\ln(s + |X(s)| + 2k)|\nabla_v f(s, X(s), V(s))|}{(s + |X(s)| + 2k)(s - |X(s)| + 2k)^2}\, ds \\ &\le \|f_0\|_1 + c \int_0^\tau \frac{\ln(2+s)}{(1+s)^3} |\nabla_v f(s, X(s), V(s))|\, ds \end{aligned}$$

by Lemmas 6.3.1 and 6.3.2. We write this for brevity as

$$|\nabla_x f(\tau)| \le \|f_0\|_1 + c \int_0^\tau I(s) |\nabla_v f(s)|\, ds. \tag{6.24}$$

Now let $Df = \partial f/\partial v_j$. Then

$$|L(Df)| = |-D\hat{v} \cdot \nabla_x f - D\hat{v} \times B \cdot \nabla_v f| \le |\nabla_x f| + |B||\nabla_v f|.$$

Integrating as above, we have

$$\begin{aligned} |\nabla_v f(\tau, X(\tau), V(\tau))| &\le \|f_0\|_1 + \int_0^\tau \{|\nabla_x f| + |B||\nabla_v f|\}(s, X(s), V(s))\, ds \\ &\le \|f_0\|_1 + \int_0^\tau |\nabla_x f|\, ds \\ &\quad + \|K\|_0 \int_0^\tau \frac{|\nabla_v f|\, ds}{(s + |X(s)| + 2k)(s - |X(s)| + 2k)}, \end{aligned}$$

where $\nabla_x f$ and $\nabla_v f$ are evaluated at $(s, X(s), V(s))$. The last kernel is at most $c(1+s)^{-2}$, as above. Since it is integrable, an application of Gronwall's inequality yields

$$|\nabla_v f(\tau)| \le c\left[\|f_0\|_1 + \int_0^\tau |\nabla_x f(s)|\, ds\right]. \tag{6.25}$$

We now have (6.24) and (6.25), where $I(s) = (1+s)^{-3}\ln(2+s)$. Substituting (6.24) into (6.25) we have

$$|\nabla_v f(\tau)| \le cc_0(1+\tau) + c^2 \int_0^\tau (\tau - s) I(s) |\nabla_v f(s)|\, ds,$$

where $c_0 = \|f_0\|_1$. Therefore

$$g(\tau) \equiv (1+\tau)^{-1}|\nabla_v f(\tau)| \le c_0 c_1 + c^2 \int_0^\tau (1+s) I(s) g(s)\, ds.$$

Since $(1+s)I(s)$ is also integrable, we deduce by Gronwall again that $g(\tau)$ is bounded. By (6.24), $|\nabla_x f|$ is also bounded for $0 \le \tau \le t$. The bounds are independent of τ, t, x and v. Putting $\tau = t$ we conclude that

$$|\nabla_x f(t,x,v)| = |\nabla_x f(t, X(t), V(t))| \le c$$

and

$$|\nabla_v f(t,x,v)| = |\nabla_v f(t, X(t), V(t))| \le c(1+t).$$

6.5. Estimates on the Fields

As we showed above, the characteristics, defined by (6.13) and (6.14), exist globally. Because $|\partial X/\partial s| = |\widehat{V}| < 1$, and $f_{\alpha 0}$ has support in $\{|x| \le k\}$, the particle density $f_\alpha(t,x,v)$ defined by (6.15) has support in $|x| \le t+k$. Therefore so do j and ρ. Therefore E^* and B^*, the solutions of Maxwell's equations with these sources, also have supports in $|x| \le t+k$. In order to prove Theorem 6.2.2, it remains to show that $\|K^*\| < \varepsilon$ for ε and ε_0 sufficiently small.

LEMMA 6.5.1 *There is a constant $c > 0$ so that*

$$\|K^*\|_0 \le c\varepsilon_0(1 + \|K\|_0)$$

for all $K \in \mathcal{K}$ provided ε is sufficiently small.

Proof. We begin with the representation formula for E^* from Chapter 5, i.e., from Theorem 3 of [15]; namely,

$$E^* = E_z^* + E_T^* + E_S^*, \tag{6.26}$$

where the terms are given explicitly as follows:

$$\begin{aligned} E_S^*(t,x) &= -\sum_\alpha e_\alpha \int_{|y-x|\le t} \int \frac{\omega + \hat{v}_\alpha}{1 + \hat{v}_\alpha \cdot \omega} S_\alpha f_\alpha \, dv \frac{dy}{|y-x|} \\ &= -\sum_\alpha e_\alpha^2 \int_{|y-x|\le t} \int \nabla_v \left[\frac{\omega + \hat{v}_\alpha}{1 + \hat{v}_\alpha \cdot \omega} \right] \cdot (E + \hat{v}_\alpha \times B) f_\alpha \, dv \frac{dy}{|y-x|}, \end{aligned}$$

where $f_\alpha = f_\alpha(t-|y-x|,y,v)$, $E = E(t-|y-x|,y)$, $B = B(t-|y-x|,y)$, $\omega = (y-x)/|y-x|$ and $S_\alpha = \partial_t + \hat{v}_\alpha \cdot \nabla_x$ from the last chapter. Next, we have

$$E_T^*(t,x) = -\sum_\alpha e_\alpha \int_{|y-x|\le t} \int \frac{(\omega+\hat{v}_\alpha)(1-|\hat{v}_\alpha|^2)}{(1+\hat{v}_\alpha\cdot\omega)^2} f_\alpha \, dv \frac{dy}{|y-x|^2}.$$

The term E_z^* is the sum of the solution $\mathcal{E}(t,x)$ of the homogeneous ("free") Maxwell equations with the same initial data and of the boundary term arising from integration by parts in y (using the fact that T_j is an exact derivative). The latter involves the expression

$$\sum_j \omega_j a_j = \frac{\omega_i - (\hat{v}_\alpha\cdot\omega)\hat{v}_{\alpha i}}{1+\hat{v}_\alpha\cdot\omega}$$

(see the proof of Theorem 5.3.1 where the expression a_j is defined). Therefore

$$(6.27)\quad E_z^*(t,x) = \mathcal{E}(t,x) - \sum_\alpha \frac{e_\alpha}{t} \int_{|y-x|=t} \int \frac{\omega - (\hat{v}_\alpha\cdot\omega)\hat{v}_\alpha}{1+\hat{v}_\alpha\cdot\omega} f_{\alpha 0}(y,v)\, dv\, dS_y,$$

where

$$(6.28)\ \mathcal{E}(t,x) = \frac{1}{4\pi t^2}\int_{|y-x|=t} [E_0(y) + ((y-x)\cdot\nabla)E_0(y) + t\nabla\times B_0(y)]\, dS_y$$
$$-\frac{1}{4\pi t}\int_{|y-x|=t} \sum_\alpha e_\alpha \int \hat{v}_\alpha f_\alpha(0,y,v)\, dv\, dS_y.$$

Now E_z^* is easy to estimate. Both of its terms have supports in $|t-|x|| \le k$ because $f_{\alpha 0}(y,v)$, $E_0(y)$ and $B_0(y)$ have supports in $|y|\le k$. Furthermore, both of them possess a factor t^{-1}. Therefore we can replace t^{-1} by $(t+|x|+2k)^{-1}$, and we can insert arbitrary factors of $t-|x|+2k$. Hence

$$\|E_z^*\|_0 \le c\,(\|E_0\|_{C^1} + \|B_0\|_{C^1} + \|f_0\|_{C^0}).$$

As for the main terms E_T^* and E_S^*, we note that the kernels are bounded because $|v|\le\beta$ (by Theorem 6.2.1) and $|\omega|=1$. Thus

$$|E_T^*(t,x)| \le c\sum_\alpha \int_{|y-x|\le t}\int f_\alpha(t-|x-y|,y,v)\, dv \frac{dy}{|y-x|^2}$$
$$\le c\varepsilon_0 \int_{|y-x|\le t} (t-|x-y|+|y|+2k)^{-3}|y-x|^{-2}\varphi\, dy,$$

where $\varphi = 1$ for $|y| < t-|x-y|+k$ and $\varphi = 0$ for $|y| > t-|x-y|+k$, since $f_\alpha(\tau,y,v) = 0$ for $|y|>\tau+k$ and $\int f_\alpha(\tau,x,v)\,dv \le c\|f_0\|_0(1+\tau)^{-3} \le c\varepsilon_0(\tau+|y|+2k)^{-3}$ by Lemma 6.4.1. Similarly

$$|E_S^*(t,x)| \le c\sum_\alpha \int_{|y-x|\le t}\int f_\alpha(t-|x-y|,y,v)\, dv |K(t-|x-y|,y)| \frac{dy}{|y-x|}$$
$$\le c\varepsilon_0\|K\|_0 \int_{|y-x|\le t} (t-|x-y|+|y|+2k)^{-4}$$
$$(t-|x-y|-|y|+2k)^{-1}|y-x|^{-1}\varphi\, dy.$$

By Lemma 6.5.2 below, we can reduce these triple integrals to double integrals:

$$|E_T^*(t,x)| \le \frac{c\varepsilon_0}{r}\int_0^t\int_a^b(\tau+\lambda+2k)^{-3}\lambda d\lambda\frac{d\tau}{t-\tau}, \tag{6.29}$$

$$|E_S^*(t,x)| \le \frac{c\varepsilon_0\|K\|_0}{r}\int_0^t\int_a^b(\tau+\lambda+2k)^{-4}(\tau-\lambda+2k)^{-1}\lambda\varphi\, d\lambda\, d\tau, \tag{6.30}$$

where $\varphi = 0$ for $\lambda > \tau + k$, $a = |r - t + \tau|$, $b = r + t - \tau$, $r = |x|$, $\lambda = |y|$ and $\tau = t - |x-y|$. It suffices to show that $|E_T^*(t,x)| \le c\varepsilon_0(t+r+2k)^{-1}(t-r+2k)^{-1}$ and $|E_S^*(t,x)| \le c\varepsilon_0\|K\|_0(t+r+2k)^{-1}(t-r+2k)^{-1}$, plus the same estimates for $B^*(t,x)$, which are very similar.

Consider the integral in (6.30). We estimate $(\tau+\lambda+2k)^{-1}\lambda \le 1$ and $(\tau-\lambda+2k)^{-1}\varphi \le k^{-1}$. Denote by

$$s(x) = \begin{cases} x & x > 0 \\ 0 & x \le 0. \end{cases}$$

Then the integral in (6.30) is less than

$$\begin{aligned}\frac{1}{k}\int_0^t\int_a^b(\tau+\lambda+2k)^{-3}\,d\lambda\,d\tau &= \frac{1}{k}\int_0^t\frac{(b-a)(b+a+2\tau+4k)}{(\tau+a+2k)^2(\tau+b+2k)^2}\,d\tau\\ &= \frac{4r}{k}\int_0^{s(t-r)}\frac{t+2k}{(t-r+2k)^2(t+r+2k)^2}\,d\tau\\ &\quad+\frac{2}{k}\int_{s(t-r)}^t\frac{(t-\tau)(2r+2\tau+4k)\,d\tau}{(2\tau+r-t+2k)^2(t+r+2k)^2}.\end{aligned}$$

The first integral is at most $cr(t-r+2k)^{-1}(t+r+2k)^{-1}$. In the second one, we use $t-\tau \le r$ and $2r+2\tau+4k \le 2(t+r+2k)$, so we get at most

$$\frac{cr}{t+r+2k}\int_{s(t-r)}^t\frac{d\tau}{(2\tau+r-t+2k)^2} \le \frac{cr}{t+r+2k}\int_{s(t-r)}^\infty\frac{d\tau}{(\tau+2k)^2},$$

which is also less than $cr(t-r+2k)^{-1}(t+r+2k)^{-1}$, since $r \le t+k$.

The integral in (6.29) is also broken up at $\tau = s(t-r)$. For $\tau \le s(t-r)$, we write $\lambda \le b = r+t-\tau \le 2(t-\tau)$ in order to reduce it to one of the integrals already estimated. On the other hand, for $\tau \ge s(t-r)$, we estimate $\lambda \le \tau+\lambda+2k$ to obtain

$$\int_{s(t-r)}^t\int_a^b\frac{d\lambda\,d\tau}{(\tau+2k)^2(t-\tau)} \le 2\int_{s(t-r)}^t\frac{d\tau}{(\tau+2k)^2} \le 2r(t-r+2k)^{-1}(t+2k)^{-1},$$

since $r \le t+k$. Thus each integral in (6.29) and (6.30) is dominated by a constant times $(r(t+r+2k)^{-1}(t-r+2k)^{-1})$. This establishes the required estimates for E^* and essentially completes the proof of Lemma 6.5.1 since the estimates for B^* are identical.

Indeed, from Theorem 3 of [15], using the methods of the last chapter, we have (for a single species, say) the representation

$$B^* = B_z^* + B_T^* + B_S^*, \tag{6.31}$$

where B_z^* is given by a formula similar to (6.27) (and hence depends only on the data), and

$$B_T^* = \iint\limits_{|y-x|\le t} \frac{(\omega\times\hat{v})(1-|\hat{v}|^2)}{(1+\hat{v}\cdot\omega)^2} f(t-|y-x|,y,v)\,dv\frac{dy}{|y-x|^2};$$

$$(B_S^*)^i = \iint\limits_{|y-x|\le t} \frac{(\omega\times\hat{v})_i}{1+\hat{v}\cdot\omega}(Sf)(t-|y-x|,y,v)\,dv\frac{dy}{|y-x|}$$

$$= \iint\limits_{|y-x|\le t} \nabla_v\left[\frac{(\omega\times\hat{v})_i}{1+\hat{v}\cdot\omega}\right]\cdot(E+\hat{v}\times B)f\,dv\frac{dy}{|y-x|}.$$

Again the kernels appearing here are bounded because $|v|\le\beta$ by Theorem 6.2.1 and $|\omega|=1$. Hence B_T^* and B_S^* are estimable exactly as in (6.29) and (6.30).

LEMMA 6.5.2 *For any continuous function $g(\tau,\lambda)$ of two real variables, and $h(\sigma)$ of one real variable,*

$$\int\limits_{|y-x|\le t} g(t-|x-y|,|y|)h(|x-y|)\,dy = \frac{2\pi}{|x|}\int_0^t\int_{||x|-t+\tau|}^{|x|+t-\tau} g(\tau,\lambda)\lambda\,d\lambda(t-\tau)h(t-\tau)\,d\tau,$$

where the integration on the left is over a ball in $\mathbb{R}^3$.

Proof. The left side can be written as

$$\int_0^t\int_{|x-y|=t-\tau} g(\tau,|y|)\,dS_y\,h(t-\tau)\,d\tau.$$

So it suffices to show that

$$\int_{|x-y|=t-\tau} g(\tau,|y|)\,dS_y = \frac{2\pi(t-\tau)}{|x|}\int_{||x|-t+\tau|}^{|x|+t-\tau} g(\tau,\lambda)\lambda\,d\lambda.$$

That is, for any function f,

$$\int_{|x-y|=\sigma} f(|y|)\,dS_y = \frac{2\pi\sigma}{|x|}\int_{||x|-\sigma|}^{|x|+\sigma}\lambda f(\lambda)\,d\lambda.$$

By symmetry we may take $x=(0,0,r)$. Then

$$\int_{|y-x|=\sigma} f(|y|)\,dS_y = \sigma^2\int_{|\omega|=1} f(|x+\sigma\omega|)\,d\omega$$

$$= 2\pi\sigma^2\int_0^\pi \sin\varphi\, f\left((r^2+\sigma^2+2r\sigma\cos\varphi)^{1/2}\right)\,d\varphi.$$

Changing variables now by $\lambda = (r^2 + \sigma^2 + 2r\sigma\cos\varphi)^{1/2}$, we obtain the desired result.

6.6. Estimates on Derivatives of the Fields

In this section we shall estimate $\|K^*\|_1$. Lemma 6.6.1 (below) together with Lemma 6.5.1 will provide the bound

$$\|K^*\| \leq c\varepsilon_0(1 + \|K\| + \|K\|^2).$$

Therefore if $\|K\| \leq \varepsilon$, if ε is sufficiently small for the validity of the previous estimates, if $\varepsilon < 1/2$ and if $\varepsilon_0 < \varepsilon/(2c)$, then $\|K^*\| < \varepsilon$. As remarked earlier, this will prove Theorem 6.2.2 and therefore Theorem 6.1.1.

LEMMA 6.6.1 *There is a constant c such that*

$$\|K^*\|_1 \leq c\varepsilon_0(1 + \|K\|_1^2).$$

Proof. We differentiate the representation formula (6.26) for E^*, repeating the technique given in the previous chapter. Thus the derivative $\partial/\partial x_k$ of the ith component of E^* is

$$\partial_k E^{*i}(t,x) = \partial_k E_z^{*i} - \sum_\alpha e_\alpha \iint \frac{(\omega_i + \hat{v}_{\alpha i})(1 - |\hat{v}_\alpha|^2)}{(1 + \hat{v}_\alpha \cdot \omega)^2} \partial_k f_\alpha \, dv \frac{dy}{|y-x|^2}$$
$$- \sum_\alpha e_\alpha^2 \iint \nabla_v \left[\frac{\omega_i + \hat{v}_{\alpha i}}{1 + \hat{v}_\alpha \cdot \omega}\right] \cdot \partial_k\{(E + \hat{v}_\alpha \times B) f_\alpha\} \, dv \frac{dy}{|y-x|}$$

integrated over $|y - x| \leq t$ and over all $v \in \mathbb{R}^3$. The derivative ∂_k is broken into T_j and S components as before. Each integration by parts in the tangential variables T_j brings in a term at the base $t = 0$ of the cone. Repeating the method of the last chapter (which is Theorem 4 of [15]), we write the result as

$$\partial_k E^{*i} = A_z + A_w + A_{TT} + A_{TS} + A_{ST} + A_{SS}. \tag{6.32}$$

The various terms are given as follows. (For notational simplicity, we drop the subscripts α and take $m_\alpha = e_\alpha = 1$).

$$A_{TT} = \iint a(\omega, \hat{v}) f \, dv \frac{dy}{|y-x|^3},$$
$$A_{TS} + A_{ST} = \iint b(\omega, \hat{v}) S f \, dv \frac{dy}{|y-x|^2}$$
$$= \iint \nabla_v b(\omega, \hat{v}) \cdot (E + \hat{v} \times B) f \, dv \frac{dy}{|y-x|^2},$$
$$A_{SS} = \iint c(\omega, \hat{v}) S^2 f \, dv \frac{dy}{|y-x|},$$

where the kernels a, b and c are found by the methods of the last chapter. The integration ranges over $|y-x| \leq t$ and $v \in \mathbb{R}^3$. The expression A_z is the sum of all the terms involving the initial data, namely, with $\omega = (y-x)/|y-x|$,

(6.33)
$$A_z = \partial_k E_z^{*i} - \frac{1}{t^2} \iint_{|y-x|=t} d(\omega, \hat{v}) f_0(y,v)\, dv\, dS_y + \frac{1}{t} \iint_{|y-x|=t} e(\omega, \hat{v}) Sf(0,y,v)\, dv\, dS_y,$$

where d and e are kernels which once again are bounded for bounded v (see the proof of Theorem 5.4.1). These integrals come from the passage from E_T^* to A_{TT}, and from E_S^* to A_{ST}, respectively. However in the passage from E_T^* to A_{TT} we also pick up the term

$$A_w = \int_{|\omega|=1} \int d(\omega, \hat{v}) f(t,x,v)\, d\omega\, dv,$$

because of the singularity $|y-x|^{-2}$ at the vertex of the cone where $y = x$. Now we shall estimate each term in (6.32).

The "vertex term" is estimated using Lemma 6.4.1 as

$$|A_w| \leq c \int f(t,x,v)\, dv \leq c\varepsilon_0 (1+t)^{-3},$$

which is more than sufficient. The "base term" A_z has several parts. One is an integral involving derivatives of E_0 and B_0 up to order two, obtained by differentiating (6.28). Since the integration is on the sphere $|y-x| = t$ and since $|y| \leq k$, it has support in $|t - |x|| \leq k$. Also, it is of the order $O(t^{-1})$. Hence it is $O((t+|x|+2k)^{-1}(t-|x|+2k)^{-2})$. The other parts of A_z are the derivatives of the last terms in (6.27), (6.28). They are $O(t^{-1})$ times an integral over $|y-x| = t$, and so are estimated in the same way as the first part. The integrals appearing in (6.33) have the same general form. In the last term in (6.33) appears the expression $Sf(0,x,v) = -(E_0 + \hat{v} \times B_0) \cdot \nabla_v f_0$, which once again has its support in $|x| \leq k$. Therefore

$$|A_z(t,x)| \leq c(t+|x|+2k)^{-1}(t-|x|+2k)^{-2}(\|E_0\|_{C^2} + \|B_0\|_{C^2} + \|f_0\|_{C^1}).$$

The kernel $a(\omega, \hat{v})$ in A_{TT} is bounded because $|v| \leq \beta$. We break A_{TT} up into two parts. The part over the "base" or "shell" $1 \leq |y-x| \leq t$ is less than

$$\begin{aligned} &c \int_{1 \leq |y-x| \leq t} \int f(t-|x-y|, y, v)\, dv \frac{dy}{|y-x|^3} \\ &\leq c\|f_0\|_0 \int_{1 \leq |y-x| \leq t} (t - |x-y| + |y| + 2k)^{-3} \frac{dy}{|y-x|^3} \\ &\leq \frac{c\varepsilon_0}{|x|} \int_0^{t-1} \int_a^b \frac{\lambda\, d\lambda}{(\tau + \lambda + 2k)^3} \frac{d\tau}{(t-\tau)^2}, \end{aligned}$$

where $a = |r - t + \tau|$, $b = r + t - \tau$ and $r = |x|$ as before, by Lemmas 6.4.1 and 6.5.2. The last integral is estimated as follows. (Assume $t > 2$ or else the estimate is trivial.) It is broken into the intervals $[0, t/2]$ and $[t/2, t-1]$:

$$\begin{aligned}\int_0^{t/2} &\leq \frac{4}{t^2}\int_0^{t/2}\int_a^b \frac{d\lambda}{(\lambda+\tau+2k)^2}\,d\tau \leq \frac{4}{t^2}\int_0^t \frac{b-a}{(a+\tau+2k)(b+\tau+2k)}\,d\tau \\ &= \frac{4}{t^2(t+r+2k)}\left\{\int_0^{s(t-r)} \frac{2r}{t-r+2k}\,d\tau + \int_{s(t-r)}^t \frac{2(t-\tau)}{2\tau+r-t+2k}\,d\tau\right\} \\ &\leq \frac{4}{t^2(t+r+2k)}\left\{2r + \frac{1}{t-r+2k}\int_{s(t-r)}^t 2(t-\tau)\,d\tau\right\} \\ &\leq c(t+r+2k)^{-2}(t-r+2k)^{-1}r,\end{aligned}$$

since $r \leq t+k$. On the other hand,

$$\begin{aligned}\int_{t/2}^{t-1} &\leq (t/2+2k)^{-3}\int_{t/2}^{t-1}\int_a^b \lambda\,d\lambda\frac{d\tau}{(t-\tau)^2} = 4r(t/2+2k)^{-3}\int_{t/2}^{t-1}\frac{d\tau}{t-\tau} \\ &\leq cr(t+1)^{-3}(1+\ln(t/2)) \leq \frac{cr\ln(t+r+2k)}{(t+r+2k)^3},\end{aligned}$$

since $r \leq t+k$. This takes care of the "base" part of A_{TT}.

Because of the singularity at $y = x$, we must use the fact that the kernel $a(\omega, \hat{v})$ has zero average on $|\omega| = 1$. Therefore the "tip" of A_{TT}, that is, the integral over $|y-x| < 1$ is equal to

$$\int_{|y-x|<1}\int a(\omega,\hat{v})[f(t-|x-y|,y,v) - f(t-|x-y|,x,v)]\,dv\frac{dy}{|y-x|^3}.$$

We use the Mean Value Theorem to represent the difference, and then use Lemma 6.4.2. We note that the integration ranges only over the set

$$\mathcal{A}(\tau) = \{v : |X(0,\tau,z,v)| \leq k \quad \text{for all} \quad z \in [x,y]\},$$

where $[x,y]$ is the line segment joining x and y and where $\tau = t - |x-y|$. By Lemma 6.4.1, this set has diameter at most $c/(\tau+2)$. Therefore the integral is bounded by

$$c\|f_0\|_1\int_{|y-x|<1}(2+t-|x-y|)^{-3}\frac{dy}{|y-x|^2} \leq \frac{c\varepsilon_0}{(1+t)^3},$$

which is more than sufficient for our purposes. This completes the estimation of A_{TT}.

Next, $A_{TS} + A_{ST}$ is bounded by

$$c\|f_0\|_0\|K\|_0\,\frac{1}{r}\int_0^t\int_a^b(\tau+\lambda+2k)^{-4}(\tau-\lambda+2k)^{-1}\lambda\varphi\,d\lambda\frac{d\tau}{t-\tau},$$

where $\varphi = 0$ for $\lambda > \tau + k$, as we did several times before by the definition of $\|K\|_0$ and by Lemmas 6.4.1 and 6.5.2. To estimate this integral, we note that

$(\tau-\lambda+2k)^{-1}\varphi \le k^{-1}$ and $\tau+\lambda+2k \ge \tau+a+2k \ge t-r+2k$. So the integral is less than

$$\frac{1}{kr(t-r+2k)}\int_0^t\int_a^b \frac{\lambda\, d\lambda}{(\tau+\lambda+2k)^3}\frac{d\tau}{t-\tau} \le c(t-r+2k)^{-2}(t+r+2k)^{-1}$$

by the previous estimate of the integral in (6.29).

It remains to estimate A_{SS}. To do this, we decompose it into five terms as in Chapter 5 (which is Eq. (66) of [15]) as

$$4\pi A_{SS} = \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV}' + \mathrm{IV}''.$$

The first is

$$\begin{aligned}
|\mathrm{I}| &= \left|\iint f(E+\hat{v}\times B)\cdot\nabla_v[\nabla_v c\cdot(E+\hat{v}\times B)]\frac{dv\,dy}{|y-x|}\right| \\
&\le c\iint f\,dv|K|^2\frac{dy}{|y-x|} \\
&\le c\|f_0\|_0\|K\|_0^2\,\frac{1}{r}\int_0^t\int_a^b \frac{\lambda\,d\lambda\,d\tau}{(\tau+\lambda+2k)^5(\tau-\lambda+2k)^2}
\end{aligned}$$

by Lemmas 6.3.2, 6.4.1 and 6.5.2 again. The second is

$$\begin{aligned}
|\mathrm{II}| &= \left|\iint f\nabla_v c\cdot S(E+\hat{v}\times B)\frac{dv\,dy}{|y-x|}\right| \\
&\le c\|f_0\|_0\|K\|_1\frac{1}{r}\int_0^t\int_a^b \frac{\ln(\tau+\lambda+2k)\lambda\,d\lambda}{(\tau+\lambda+2k)^4(\tau-\lambda+2k)^2}.
\end{aligned}$$

The integral in I is less than the integral in II, so it suffices to estimate the latter. In the integral in II we replace $\ln(\tau+\lambda+2k)$ by $\ln(t+r+2k)$ and λ by $\tau+\lambda+2k$ in order to obtain the upper bound

$$\begin{aligned}
&\ln(t+r+2k)\int_0^t\int_a^b \frac{d\lambda\,d\tau}{(\tau+\lambda+2k)^3(\tau-\lambda+2k)^2} \\
&\le \frac{1}{2}\ln(t+r+2k)\int_{|t-r|}^{t+r}\frac{d\alpha}{(\alpha+2k)^3}\int_{-k}^{\infty}\frac{d\beta}{(\beta+2k)^2} \\
&= c\ln(t+r+2k)\{(|t-r|+2k)^{-2}-(t+r+2k)^{-2}\}.
\end{aligned}$$

We replace $|t-r|$ by $t-r$ to obtain at most

$$\begin{aligned}
&4cr(t+2k)(t-r+2k)^{-2}(t+r+2k)^{-2}\ln(t+r+2k) \\
&\le 4cr(t-r+2k)^{-2}(t+r+2k)^{-1}\ln(t+r+2k)
\end{aligned}$$

as desired.

The third term in A_{SS} is

$$\mathrm{III} = \iint \hat{c}_{j\ell} f\partial_j K^\ell\frac{dv\,dy}{|y-x|},$$

where $\hat{c}_{j\ell} = (\delta_{j\ell} - \hat{v}_j\hat{v}_\ell)(1+|v|^2)^{-1/2}c(\omega, \hat{v})$ is bounded. Hence III is estimated exactly as II was. Next, the fourth term

$$|\mathrm{IV}'| = \left| - \iint \hat{c}_{j\ell} K^\ell \frac{\omega_j}{1+\hat{v}\cdot\omega} \nabla_v \cdot (fK) \frac{dv\,dy}{|y-x|} \right| \le c \iint f\,dv |K|^2 \frac{dy}{|y-x|}$$

is estimated exactly as I was. The fifth term is

$$\mathrm{IV}'' = \iint \hat{c}_{j\ell} K^\ell \left[\delta_{jp} - \frac{\omega_j \hat{v}_p}{1+\hat{v}\cdot\omega} \right] T_p f \frac{dv\,dy}{|y-x|}.$$

As above, the integration ranges over the ball $|y-x| \le t$. We integrate by parts in T_p, obtaining on the one hand the term on $|y-x| = t$,

$$\frac{1}{t} \iint_{|y-x|=t} \hat{c}_{j\ell}\omega_j (1+\hat{v}\cdot\omega)^{-1} K^\ell(0,y)\, f_0(y,v)\, dv\, dS_y,$$

which is treated just like the integrals in A_z. On the other hand, we also obtain terms bounded by

$$c \iint_{|y-x|\le t} f |\nabla_x K| \frac{dv\,dy}{|y-x|} + c \iint_{|y-x|\le t} f|K| \frac{dv\,dy}{|y-x|^2}.$$

Of these resulting integrals, the first one is estimated just as II was, and the second one is estimated just as $A_{TS} + A_{ST}$ was. This completes the proof of Lemma 6.6.1.

We conclude this chapter with a brief description of the generalization of this result to the "nearly neutral" case. Details can be found in [10]. In this reference the same regularity and support properties are imposed on the initial data. Moreover, the same smallness conditions are imposed on the initial fields. However, the individual plasma densities may be "large", provided that they "almost cancel" in the following sense. Let

$$g_\alpha(t,x,v) = f_\alpha(t,x,m_\alpha v), \quad g_\alpha^0(x,v) = f_\alpha^0(x, m_\alpha v).$$

Then the smallness condition on the plasma densities may be expressed by saying that the expression

$$\sum_\alpha e_\alpha m_\alpha^3 g_\alpha^0$$

is small in the C^1 sense. Thus, cancellation is allowed, whereas the result in the current chapter requires that each individual plasma density be small.

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Chapter 7

VELOCITY AVERAGES: WEAK SOLUTIONS TO THE VLASOV–MAXWELL SYSTEM

7.1. Sketch of the Problem

For many nonlinear problems, one uses the following steps. Firstly, a convenient approximation is chosen. This approximation scheme should satisfy the dual criteria that the modified problem is "easily" solvable, and that it retain the expected a priori bounds. Then one uses compactness to pass to the limit (in the sense of distributions) in the modified problem. What follows appeared originally in DiPerna–Lions [3]; we also use the approach in Kruse [14].

Let compatible data $f_0 \geq 0$, E_0, B_0 be prescribed at $t = 0$. Let δ_ϵ be the standard Friedrichs mollifier. We will consider two approximation schemes. Firstly, consider the modified system

$$(MVM) \quad f_t + v \cdot \nabla_x f + (E + v \times B) \cdot \nabla_v f = 0, \tag{7.1}$$

$$\partial_t E = c\nabla \times B - j^\epsilon \qquad \nabla \cdot E = \rho$$
$$\partial_t B = -c\nabla \times E \qquad \nabla \cdot B = 0.$$

Below the function $\nu(x)$ represents a given neutralizing background density, and $\rho = 4\pi \int f\,dv - \nu(x)$; $j = 4\pi \int vf\,dv$; $\quad j^\epsilon = \delta_\epsilon * j$. By the work of Horst [11], an initial-value problem for this system possesses global smooth solutions for fixed $\epsilon > 0$. Let (f^n, E^n, B^n) be such an approximate solution corresponding to δ_n. (We abuse notation with the mollifier). We have

$$\iint (f^n)^2\,dv\,dx = \text{ const.}, \tag{7.2}$$

$$\iint |v|^2 f^n\,dv\,dx + \int (|E^n|^2 + |B^n|^2)\,dx = \text{const.}$$

Assuming $f^n(0, x, v) \geq 0$, we have by weak compactness a subsequence $\{n_k\}$ such that

$$E^{n_k} \rightharpoonup E, \; B^{n_k} \rightharpoonup B \quad \text{in} \quad L^2((0,T) \times \mathbb{R}^3); \tag{7.3}$$

$$f^{n_k} \rightharpoonup f \quad \text{in} \quad L^2((0,T) \times \mathbb{R}^3 \times B_R). \tag{7.4}$$

Thus $\partial_t f^n \rightharpoonup \partial_t f$ in $\mathcal{D}'((0,\infty)\times\mathbb{R}^3\times B_R)$, etc. So we can easily pass to the limit in the first two terms of the Vlasov equation. The same remark is valid for the first–derivative terms in the Maxwell system.

In order to study the convergence of the nonlinear term in (MVM), let $\phi \in \mathcal{D}((0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3)$. We want to examine

$$\begin{aligned}\text{(7.5)}\qquad &\iiint \phi(E^n + v\times B^n)\cdot\nabla_v f^n\,dv\,dx\,dt\\ &= -\iiint \nabla_v\phi\cdot(E^n+v\times B^n)f^n\,dv\,dx\,dt.\end{aligned}$$

Here we are faced with a familiar problem, namely of passing to a weak limit in a product expression, each factor of which converges weakly. If we take $\phi = \tilde{\phi}(t,x)\Psi(v)$ (which is enough by density) we need convergence of e.g.,

$$\text{(7.6)}\qquad \iint \tilde{\phi}(t,x)E^n(t,x)\cdot\int \psi(v)f^n\,dv\,dx\,dt \qquad \text{for}\quad \psi\in\mathcal{D}(\mathbb{R}^3).$$

Here $\psi = \nabla_v\Psi$. If we knew **strong convergence**

$$\int f^n(\cdot,\cdot,v)\psi(v)\,dv \to \int f(\cdot,\cdot,v)\psi(v)\,dv \quad \text{in} \quad L^2((0,T)\times B_R)$$

for arbitrary T, R, we are done. This is exactly a consequence of the velocity averaging smoothing mechanism, cf. [7], [6] and [3]. We will deduce that $j^n \to j$ in $\mathcal{D}'$ from strong L^1–convergence

$$\text{(7.7)}\qquad \int f^n(\cdot,\cdot,v)\psi(v)\,dv \to \int f(\cdot,\cdot,v)\psi(v)\,dv \qquad \text{in}\quad L^1((0,T)\times B_R)$$

for continuous ψ with $\psi(v) = o(|v|^2)$ as $|v|\to\infty$.

There is another regularization, due to Lions and DiPerna [3]. We take $f_0^\epsilon \ge 0$, $f_0^\epsilon \in \mathcal{D}(\mathbb{R}^6)$ such that

$$\text{(7.8)}\qquad \iint_{\mathbb{R}^6} [|f_0 - f_0^\epsilon|(1+|v|^2) + |f_0 - f_0^\epsilon|^2]\,dv\,dx \to 0 \qquad \text{as}\quad \epsilon\to 0.$$

Take E_0^ϵ, $B_0^\epsilon \in \mathcal{D}(\mathbb{R}^3)$ with

$$\begin{aligned}&\|E_0 - E_0^\epsilon\|_2 \to 0,\\ &\|B_0 - B_0^\epsilon\|_2 \to 0.\end{aligned}$$

Consider (with $c=1$)

$$\begin{aligned}\text{(7.9)}\qquad \partial_t f^\epsilon + v\cdot\nabla_x f^\epsilon + (E^\epsilon + v\times B^\epsilon)\cdot\nabla_v f^\epsilon &= 0,\\ \partial_t E^\epsilon - \nabla\times B^\epsilon + \epsilon(-\Delta)^3E^\epsilon &= -j^\epsilon,\\ \partial_t B^\epsilon + \nabla\times E^\epsilon + \epsilon(-\Delta)^3 B^\epsilon &= 0;\end{aligned}$$

$$f^\epsilon\big|_{t=0} = f_0^\epsilon, \qquad E^\epsilon\big|_{t=0} = E_0^\epsilon, \qquad B^\epsilon\big|_{t=0} = B_0^\epsilon.$$

Then $f^\epsilon \geq 0$, $\int\int |f^\epsilon(t,x,v)|^p \, dv\, dx = \int |f_0^\epsilon|^p \, dv\, dx$ for any p, $1 \leq p \leq \infty$, and

$$\frac{d}{dt}\Big[\int_{\mathbb{R}^6} |v|^2 f^\epsilon \, dv\, dx + \int_{\mathbb{R}^3} (|E^\epsilon|^2 + |B^\epsilon|^2)\, dx\Big] + \epsilon \int_{\mathbb{R}^3} (|D^3 E^\epsilon|^2 + |D^3 B^\epsilon|^2)\, dx = 0. \tag{7.10}$$

Thus the same basic bounds hold.

7.2. The Velocity Averaging Smoothing Effect

The first such results (cf. [1], [6], [7]) asserted this: if $f \in L^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$ is a solution of

$$\partial_t f + v \cdot \nabla_x f = g \in L^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3) \tag{7.11}$$

then for $\psi \in \mathcal{D}(\mathbb{R}^3)$, $\int f(t,x,v)\psi(v)\, dv \in H^{1/2}(\mathbb{R} \times \mathbb{R}^3)$. Thus averaging in velocity improves regularity in the space and time variables. For Vlasov–Maxwell we need g to be a v–derivative. If $g = \sum_{|\alpha| \leq m} D_v^\alpha g_\alpha$ with $g_\alpha \in L^2(\mathbb{R} \times \mathbb{R}^3 \times \mathbb{R}^3)$, then the conclusion is

$$\int f(t,x,v)\psi(v)\, dv \in H^s(\mathbb{R} \times \mathbb{R}^3) \quad \text{where} \quad s = \frac{1}{2(1+m)}. \tag{7.12}$$

THEOREM 7.2.1 *Let $m \in \mathbb{N}$, $R \in (0,\infty)$, $\psi \in \mathcal{D}(B_R)$. Let $f \in L^2(\mathbb{R} \times \mathbb{R}^3 \times B_R)$ satisfy*

$$f_t + v \cdot \nabla_x f = \sum_{|\alpha| \leq m} D_v^\alpha g_\alpha \qquad \text{in} \quad \mathcal{D}'(\mathbb{R} \times \mathbb{R}^3 \times B_R) \tag{7.13}$$

where $g_\alpha \in L^2(\mathbb{R} \times \mathbb{R}^3 \times B_R)$ for all α, $|\alpha| \leq m$. Then there exists a constant $c > 0$ depending on m, R, ψ such that

$$\int_{B_R} f(\cdot,\cdot,v)\psi(v)\, dv \in H^s(\mathbb{R} \times \mathbb{R}^3), \qquad s = \frac{1}{2(1+m)}, \tag{7.14}$$

and

(7.15)

$$\left\| \int_{B_R} f(\cdot,\cdot,v)\psi(v)\, dv \right\|_{H^s} \leq c \left[\|f\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times B_R)} + \sum_{|\alpha| \leq m} \|g_\alpha\|_{L^2(\mathbb{R}\times\mathbb{R}^3\times B_R)} \right].$$

Proof. We will give the proof in the case of $m = 1$ only. Extend f to 0 outside of $[0,T]$. Let (τ,ξ) be the dual variables of the Fourier Transform $f \mapsto \hat{f}$. Then

$$\begin{aligned} \left\| \int_{B_R} f(\cdot,\cdot,v)\psi(v)\, dv \right\|_{H^s}^2 &\leq c \int\int \left| \int \hat{f}(\tau,\xi,v)\psi(v)\, dv \right|^2 (1 + |\tau|^{2s} + |\xi|^{2s})\, d\xi\, d\tau \\ &\leq \|\psi\|_2^2 \|\hat{f}\|_2^2 + \int\int \left| \int \hat{f}(\tau,\xi,v)\psi(v)\, dv \right|^2 (|\tau|^{2s} + |\xi|^{2s})\, d\xi\, d\tau. \end{aligned} \tag{7.16}$$

Define

$$I(\tau,\xi) = \int \hat{f}(\tau,\xi,v)\psi(v)\,dv \in L^2(\mathbb{R}\times\mathbb{R}^3). \tag{7.17}$$

Let $\zeta \in \mathcal{D}(\mathbb{R})$; $\zeta \equiv 1$ on $[-1,1]$; supp $\zeta \subseteq (-2,2)$; $0 \le \zeta \le 1$. We split the integral I as follows:

$$\begin{aligned} I &= \int \hat{f}(\tau,\xi,v)\psi(v)\zeta\left(\frac{\tau+v\cdot\xi}{K}\right)dv + \int \hat{f}(\tau,\xi,v)\psi(v)\left[1-\zeta\left(\frac{\tau+v\cdot\xi}{K}\right)\right]dv \\ &\equiv I_1 + I_2. \end{aligned}$$

This comes from the Fourier Transform of the equation (7.13):

$$(i\tau + iv\cdot\xi)\hat{f}(\tau,\xi,v) = \sum_{|\alpha|\le 1} D_v^\alpha \hat{g}_\alpha. \tag{7.18}$$

The parameter K will be chosen later.

For I_1 we have

$$\begin{aligned} |I_1| &\le \left(\int |\hat{f}(\tau,\xi,v)|^2\,dv\right)^{1/2}\left(\int \psi^2(v)\zeta^2\left(\frac{\tau+v\cdot\xi}{K}\right)dv\right)^{1/2} \\ &\le \|\psi\|_\infty \|\hat{f}(\tau,\xi,\cdot)\|_2 \left(\int_{B_R}\zeta^2\left(\frac{\tau+v\cdot\xi}{K}\right)dv\right)^{1/2} \end{aligned} \tag{7.19}$$

The ζ–integral I_ζ is less than

$$\int_{B_R}\chi_{[-2,2]}\left(\frac{\tau+v\cdot\xi}{K}\right)dv. \tag{7.20}$$

The integrand is nonzero for $\left|\frac{\tau+v\cdot\xi}{K}\right| \le 2$, i.e., for $-2K-\tau \le v\cdot\xi \le 2K-\tau$. Therefore

$$I_\zeta \le \int_{B_R}\chi_{[-2K-\tau,2K-\tau]}(v\cdot\xi)\,dv. \tag{7.21}$$

We can assume that $|\tau| < R|\xi|+2K$. Indeed, let $v\in B_R$. Then $|\tau| > R|\xi|+2K$ implies $|\tau| > |v||\xi|+2K \ge |v\cdot\xi|+2K \ge |\tau|-|\tau+v\cdot\xi|+2K$, i.e., $|\tau+v\cdot\xi| > 2K$, and $\zeta(\cdots)\equiv 0$ there. Thus we can write (with $\lambda = |v||\xi|\cos\theta$)

$$\begin{aligned} I_\zeta &\le \chi_{|\tau|\le R|\xi|+2K}\int_{B_R}\chi_{[-2K-\tau,2K-\tau]}(v\cdot\xi)\,dv \\ &= \chi_{|\tau|\le R|\xi|+2K}\cdot\int_0^R\int_0^\pi 2\pi\chi_{[-2K-\tau,2K-\tau]}(|v||\xi|\cos\theta)\sin\theta\,d\theta|v|^2\,d|v| \\ &\le 2\pi R\cdot\chi_{|\tau|\le R|\xi|+2K}|\xi|^{-1}\int_{-R|\xi|}^{R|\xi|}\chi_{[-2K-\tau,2K-\tau]}(\lambda)\,d\lambda \\ &\le c_R K|\xi|^{-1}\chi_{|\tau|\le R|\xi|+2K}. \end{aligned}$$

Hence

$$|I_1| \leq c\|\hat{f}(\tau,\xi,\cdot)\|_2 \cdot \chi_{|\tau|\leq R|\xi|+2K} \cdot \left(\frac{K}{|\xi|}\right)^{1/2}. \tag{7.22}$$

For I_2, we use the equation (7.18) and integrate by parts:

$$\begin{aligned} I_2 &= \int \hat{f}(\tau,\xi,v)\psi(v)[1-\zeta]\,dv = -i\sum_{|\alpha|\leq 1}\int \frac{D_v^\alpha \hat{g}_\alpha}{\tau+v\cdot\xi}\psi(v)(1-\zeta)\,dv \\ &= -i\sum_{|\alpha|\leq 1}\int(-1)^{|\alpha|}\int \hat{g}_\alpha(\tau,\xi,v)D_v^\alpha \underbrace{\left[\frac{\psi(v)\left(1-\zeta\left(\frac{\tau+v\cdot\xi}{K}\right)\right)}{\tau+v\cdot\xi}\right]}_{H}\,dv. \end{aligned}$$

Thus with $|\hat{g}| \equiv \max|\hat{g}_\alpha|$, $\quad |\alpha| = 0,\ 1$,

$$|I_2| \leq c\|\hat{g}(\tau,\xi,\cdot)\|_2\left(\|D_v H\|_2 + \|H\|_2\right).$$

Now by direct computation,

$$\begin{aligned} |D_v H| &\leq c\left[\frac{|D_v\psi||1-\zeta|}{|\tau+v\cdot\xi|} + \frac{\frac{|\xi|}{K}|D\zeta|\cdot|\psi|}{|\tau+v\cdot\xi|} + \frac{|\psi||1-\zeta|}{|\tau+v\cdot\xi|^2}|\xi|\right] \\ &\leq c\|\psi\|_{1,\infty}\cdot\left[\frac{\chi_{B_R}(v)(1-\zeta)}{|\tau+v\cdot\xi|} + \frac{|\xi|}{K}\chi_{B_R}(v)\chi_{|\tau+v\cdot\xi|\geq K}|\tau+v\cdot\xi|^{-1}\right. \\ &\qquad \left. + \frac{|\xi|\chi_{B_R}(v)}{|\tau+v\cdot\xi|^2}\chi_{|\tau+v\cdot\xi|\geq K}(1-\zeta)\right] \\ &\leq c\chi_{B_R}(v)\chi_{|\tau+v\cdot\xi|\geq K}|\tau+v\cdot\xi|^{-1}\left(1+\frac{|\xi|}{K}\right). \end{aligned}$$

For H itself we have $|H| \leq \frac{\chi_{B_R}(v)|\psi||1-\zeta|}{|\tau+v\cdot\xi|}$ as in the first term above. Thus for $|\alpha| = 0$ or 1 we have

$$\begin{aligned} \|D_v^\alpha H\|_2 &\leq c\left(1+\frac{|\xi|}{K}\right)\left(\int_{B_R}\chi_{|\tau+v\cdot\xi|\geq K}\frac{dv}{|\tau+v\cdot\xi|^2}\right)^{1/2} \\ &= c\left(1+\frac{|\xi|}{K}\right)\left(\int_{B_R}\chi_{(-\infty,-K)\cup(K,\infty)}(\tau+v\cdot\xi)\frac{dv}{|\tau+v\cdot\xi|^2}\right)^{1/2} \\ &\qquad \text{(switch to spherical coordinates, and then put } \lambda = |v||\xi|\cos\theta) \\ &= c_R\left(1+\frac{|\xi|}{K}\right)|\xi|^{-1/2}\left(\int_{-R|\xi|}^{R|\xi|}\chi_{(-\infty,-K)\cup(K,\infty)}(\tau+\lambda)\frac{d\lambda}{(\lambda+\tau)^2}\right)^{1/2} \\ &= c|\xi|^{-1/2}\left(1+\frac{|\xi|}{K}\right)\left(\int_{\tau-R|\xi|}^{\tau+R|\xi|}\chi_{(-\infty,-K)\cup(K,\infty)}(\nu)\frac{d\nu}{\nu^2}\right)^{1/2} \\ &\leq c|\xi|^{-1/2}\left(1+\frac{|\xi|}{K}\right)\left(\int_{-\infty}^{-K}\nu^{-2}\,d\nu + \int_K^\infty \nu^{-2}\,d\nu\right)^{1/2} \\ &\leq c(K|\xi|)^{-1/2}\left(1+\frac{|\xi|}{K}\right). \end{aligned}$$

This bound can be improved on the set where $(1-\zeta) \neq 0$: if we also assume that $|\tau| > R|\xi| + 2K > R|\xi|$, then we have

$$\begin{aligned}(7.23)\qquad \|D_v H\|_2 &\le c|\xi|^{-1/2}\left(1+\frac{|\xi|}{K}\right)\left(\int_{\tau-R|\xi|}^{\tau+R|\xi|} \nu^{-2}\,d\nu\right)^{1/2}\\ &= c|\xi|^{-1/2}\left(1+\frac{|\xi|}{K}\right)\left[\frac{1}{\tau-R|\xi|}-\frac{1}{\tau+R|\xi|}\right]^{1/2}\\ &= cR^{1/2}\left(1+\frac{|\xi|}{K}\right)\frac{1}{(\tau^2-R^2|\xi|^2)^{1/2}}.\end{aligned}$$

Therefore

$$|I_2| \le c\|\hat g(\tau,\xi,\cdot)\|_2\left(1+\frac{|\xi|}{K}\right)\left[\chi_{|\tau|\le R|\xi|+2K}(K|\xi|)^{-1}+\chi_{|\tau|>R|\xi|+2K}(\tau^2-R^2|\xi|^2)^{-1}\right]^{1/2}.$$

It follows that

$$\begin{aligned}|I| &\le |I_1|+|I_2|\\ &\le c(\|\hat f(\tau,\xi,\cdot)\|_2+\|\hat g(\tau,\xi,\cdot)\|_2)\cdot\\ &\quad\left[\left\{\frac{K}{|\xi|}+\frac{1}{K|\xi|}\left(1+\frac{|\xi|}{K}\right)^2\right\}\chi_{|\tau|\le R|\xi|+2K}+\chi_{|\tau|>R|\xi|+2K}\left(1+\frac{|\xi|}{K}\right)^2\frac{1}{\tau^2-R^2|\xi|^2}\right]^{1/2}.\end{aligned}$$

When $|\xi|<1$ we take $K=1$. Then

(7.24)
$$|I| \le c\left(\|\hat f(\tau,\xi,\cdot)\|_2+\|\hat g(\tau,\xi,\cdot)\|_2\right)\left[|\xi|^{-1}\chi_{|\tau|\le R|\xi|+2}+\frac{\chi_{|\tau|>R|\xi|+2}}{\tau^2-R^2|\xi|^2}\right]^{1/2}.$$

When $|\xi|>1$, we take $K=|\xi|^{1/2}$. Then

(7.25)
$$|I| \le c\left(\|\hat f(\tau,\xi,\cdot)\|_2+\|\hat g(\tau,\xi,\cdot)\|_2\right)\left[|\xi|^{-1/2}\chi_{|\tau|\le R|\xi|+2K}+\frac{(1+|\xi|)}{\tau^2-R^2|\xi|^2}\chi_{|\tau|>R|\xi|+2K}\right]^{1/2}.$$

Recall that $I(\tau,\xi)=\int \hat f(\tau,\xi,v)\psi(v)\,dv$, and that $s=1/4$ when $m=1$. Thus

$$\begin{aligned}(7.26)\qquad &\left\|\int_{B_R} f(\cdot,\cdot,v)\psi(v)\,dv\right\|_{H^{1/4}}^2\\ &\le \|\psi\|_2^2\|\hat f\|_2^2+\underbrace{\iint\left|\int \hat f(\tau,\xi,v)\psi(v)\,dv\right|^2(|\tau|^{1/2}+|\xi|^{1/2})\,d\xi\,d\tau}_{J}.\end{aligned}$$

Write $J=J_1+\cdots+J_5$, where

$$\begin{aligned}J_1 &= \iint |I(\tau,\xi)|^2|\tau|^{1/2}\chi_{|\xi|>1}\,d\xi\,d\tau;\\ J_2 &= \iint |I(\tau,\xi)|^2|\tau|^{1/2}\chi_{|\xi|\le 1}\chi_{|\tau|>R+2}\,d\xi\,d\tau;\\ J_3 &= \iint |I(\tau,\xi)|^2|\tau|^{1/2}\chi_{|\xi|\le 1}\chi_{|\tau|\le R+2}\,d\xi\,d\tau;\\ J_4 &= \iint |I(\tau,\xi)|^2|\xi|^{1/2}\chi_{|\xi|>1}\,d\xi\,d\tau;\\ J_5 &= \iint |I(\tau,\xi)|^2|\xi|^{1/2}\chi_{|\xi|\le 1}\,d\xi\,d\tau.\end{aligned}$$

Abbreviate $\Phi(\tau,\xi) = \|\hat{f}(\tau,\xi,\cdot)\|_2 + \|\hat{g}(\tau,\xi,\cdot)\|_2$. Now

$$J_1 \le c \iint \Phi^2(\tau,\xi)\chi_{|\xi|>1}\Big[\chi_{|\tau|\le R|\xi|+2K}|\tau|^{1/2}|\xi|^{-1/2} + \chi_{|\tau|>R|\xi|+2K}|\tau|^{1/2}\frac{(1+|\xi|)}{\tau^2 - R^2|\xi|^2}\Big]\, d\xi\, d\tau. \tag{7.27}$$

Recall that $K = |\xi|^{1/2}$ when $|\xi| > 1$. In the first term, $|\tau| \le R|\xi| + 2|\xi|^{1/2} \le (R+2)|\xi|$ in $|\xi| > 1$. Thus $|\tau|^{1/2}|\xi|^{-1/2} \le c_R$. In the second term,

$$\begin{aligned}
\frac{|\tau|^{1/2}(1+|\xi|)}{\tau^2 - R^2|\xi|^2} &= \frac{1+|\xi|}{|\tau|^{3/2} - R^2|\xi|^2|\tau|^{-1/2}} \\
&\le \frac{1+|\xi|}{(R|\xi|+2K)^{3/2} - R^2|\xi|^2(R|\xi|+2K)^{-1/2}} \\
&= \frac{1+|\xi|}{(R|\xi|+2K)^{-1/2}[R^2|\xi|^2 + 4RK|\xi| + 4K^2 - R^2|\xi|^2]} \\
&\le \frac{1+|\xi|}{(R+2)^{-1/2}|\xi|^{-1/2}\cdot 4RK|\xi|} \\
&\le c_R.
\end{aligned}$$

Thus

$$J_1 \le c_R \iint \Phi^2(\tau,\xi)\, d\xi\, d\tau. \tag{7.28}$$

For J_2 we have

$$J_2 \le c \iint \Phi^2(\tau,\xi)\chi_{|\xi|\le 1}\chi_{|\tau|>R+2}\Big[\chi_{|\tau|\le R|\xi|+2}|\xi|^{-1}|\tau|^{1/2} + \chi_{|\tau|>R|\xi|+2}\frac{|\tau|^{1/2}}{\tau^2 - R^2|\xi|^2}\Big]\, d\xi\, d\tau. \tag{7.29}$$

The first term vanishes because $R + 2 < |\tau| \le R|\xi| + 2$ which implies $|\xi| > 1$, a contradiction. For the second term, we have

$$\frac{|\tau|^{1/2}}{\tau^2 - R^2|\xi|^2} \le \frac{|\tau|^{1/2}}{\tau^2 - (|\tau|-2)^2} = \frac{|\tau|^{1/2}}{4|\tau| - 4} = \frac{|\tau|^{1/2}}{2|\tau| + 2(|\tau|-2)} \le c.$$

Thus

$$J_2 \le c \iint \Phi^2\, d\xi\, d\tau. \tag{7.30}$$

For J_3 we have simply

$$\begin{aligned}
|J_3| &= \iint |I(\tau,\xi)|^2|\tau|^{1/2}\chi_{|\xi|\le 1}\chi_{|\tau|\le R+2}\, d\xi\, d\tau \\
&\le (R+2)^{1/2} \iint \left|\int \hat{f}(\tau,\xi,v)\psi(v)\, dv\right|^2 d\xi\, d\tau \\
&\le (R+2)^{1/2}\|\psi\|_2^2\|\hat{f}\|_2^2 \le c\|\hat{f}\|_2^2.
\end{aligned} \tag{7.31}$$

In J_4 we have $|\xi| > 1$, hence

$$J_4 \leq c \iint \Phi^2(\tau,\xi)\chi_{|\xi|>1}\left[1 + \chi_{|\tau|>R|\xi|+2K}\frac{(|\xi|^{1/2}+|\xi|^{3/2})}{\tau^2 - R^2|\xi|^2}\right] d\xi\, d\tau. \tag{7.32}$$

Now

$$\begin{aligned} \frac{|\xi|^{1/2}+|\xi|^{3/2}}{\tau^2-R^2|\xi|^2} &\leq \frac{|\xi|^{1/2}+|\xi|^{3/2}}{(R|\xi|+2K)^2-R^2|\xi|^2} \\ &= \frac{|\xi|^{1/2}+|\xi|^{3/2}}{4R|\xi|K+4K^2} \\ &\leq \frac{|\xi|^{1/2}+|\xi|^{3/2}}{4RK|\xi|} \\ &\leq c_R \end{aligned} \tag{7.33}$$

because $K = |\xi|^{1/2}$ in this case.

Lastly, for J_5 we write

$$J_5 = \iint |I(\tau,\xi)|^2|\xi|^{1/2}\chi_{|\xi|\leq 1}\, d\xi\, d\tau \leq c\|\hat{f}\|_2^2 \qquad \text{as in } J_3. \tag{7.34}$$

Collecting all such terms, we get

$$\left\| \int_{B_R} f(\cdot,\cdot,v)\psi(v)\, dv \right\|_{H^{1/4}(\mathbb{R}\times\mathbb{R}^3)} \leq c(\|\hat{f}\|_2^2 + \|\hat{g}\|_2^2)^{1/2}. \tag{7.35}$$

This proves Theorem 7.2.1.

7.3. Convergence of the Current Density

THEOREM 7.3.1 *Let*

$$\begin{aligned} f^n &\in C^1([0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3) \\ E^n, B^n &\in C^0([0,\infty)\times\mathbb{R}^3) \end{aligned}$$

be classical solutions of (MVM) such that
(i) $f^n \geq 0$ for all n,
(ii) $\iint (f^n)^2\, dv\, dx \leq c_2$ for all n,
(iii) $\iint |v|^2 f^n\, dv\, dx + \int(|E^n|^2 + |B^n|^2)\, dx \leq c_3$ for all n.

For $T > 0$ arbitrary let $f^n \rightharpoonup f$ in $L^2((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$ through a subsequence $\{n_k\}$. Let $\psi \in C^0(\mathbb{R}^3)$ satisfy $|\psi(v)| = o(|v|^2)$ as $|v| \to \infty$. Then for every $S < \infty$,

$$\int f^{n_k}(\cdot,\cdot,v)\psi(v)\, dv \to \int f(\cdot,\cdot,v)\psi(v)\, dv \text{ in } L^1((0,T), B_S). \tag{7.36}$$

Proof. $f \geq 0$ a.e. because for all bounded measurable sets $A \subset (0,\infty)\times\mathbb{R}^3\times\mathbb{R}^3$,

$$(7.37)\qquad \int_A f\,dv\,dx\,dt = \lim_{k\to\infty}\int_A f^{n_k}\,dv\,dx\,dt \geq 0.$$

Similarly, for any measurable set $A \subset (0,T)$,

$$(7.38)\qquad \begin{aligned}\iiint\limits_A f|v|^2\,dv\,dx\,dt &= \lim_{p\to\infty}\int_A\int_{B_p}\int_{B_p} f|v|^2\,dv\,dx\,dt.\\ &= \lim_{p\to\infty}\lim_{k\to\infty}\int_A\int_{B_p}\int_{B_p} f^{n_k}|v|^2\,dv\,dx\,dt\\ &\leq c_3|A|.\end{aligned}$$

Hence $\iint f|v|^2\,dv\,dx \leq c_3$ a.e. $t \in (0,T)$, and thus

$$(7.39)\qquad \begin{aligned}\int_0^T \iint\limits_{B_S} f|\psi|\,dv\,dx\,dt &= \int_0^T \iint\limits_{B_S B_1} f|\psi|\,dv\,dx\,dt + \int_0^T \iint\limits_{B_S B_1^c} f\frac{|\psi|}{|v|^2}|v|^2\,dv\,dx\,dt\\ &\leq \sup_{|v|\leq 1}|\psi(v)|\|f\|_{L^2((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)}\cdot cT^{1/2}|B_S|^{1/2}\\ &\qquad + \sup_{|v|\geq 1}\frac{|\psi(v)|}{|v|^2}\cdot c_3T\\ &< \infty.\end{aligned}$$

Thus by Fubini's Theorem, the integral $\int f(\cdot,\cdot,v)\psi(v)\,dv$ belongs to $L^1((0,T)\times B_S)$.

To get the L^1-convergence we proceed with several steps.

Step 1: Let $\psi \in \mathcal{D}$; assume also that f^n is bounded in $L^\infty((0,T)\times\mathbb{R}^3\times\mathbb{R}^3)$. We have $\partial_t f^n + v\cdot\nabla_x f^n = \nabla_v \cdot g_2^n$, where

$$(7.40)\qquad g_2^n = -(E^n + v\times B^n)f^n.$$

Take $\zeta_\delta \in \mathcal{D}(\mathbb{R})$, $\zeta_\delta = 1$ on $[\delta,T]$, $\operatorname{supp}\zeta_\delta \subset \left(\frac{\delta}{2},2T\right)$, $0\leq\zeta\leq 1$. Put

$$f_\delta^n = \zeta_\delta(t)f^n,\quad g_{1,\delta}^n = \zeta_\delta'(t)f^n,\quad g_{2,\delta}^n = \zeta_\delta(t)g_2^n.$$

Then

$$\partial_t f_\delta^n + v\cdot\nabla_x f_\delta^n = g_{1,\delta}^n + \nabla_v\cdot g_{2,\delta}^n$$

holds in $\mathcal{D}'$. By hypothesis, $f_\delta^n, g_{1,\delta}^n$ are bounded in $L^2(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^3)$. Moreover, $g_{2,\delta}^n$ is bounded in $L^2(\mathbb{R}\times\mathbb{R}^3\times B_R)$ for all $R>0$, because by our additional assumption

$$(7.41)\qquad \begin{aligned}\iiint_{B_R}(g_{2,\delta}^n)^2\,dv\,dx\,dt &\leq \|f^n\|_\infty^2\int_{\delta/2}^{2T}\int\int_{B_R}|E^n + v\times B^n|^2\,dv\,dx\,dt\\ &\leq c_R\cdot c_3T\|f^n\|_\infty^2.\end{aligned}$$

Take R such that $\operatorname{supp}\psi \subset B_R$. We know from Theorem 7.2.1 that

$$\left\| \int f_\delta^n(\cdot,\cdot,v)\psi(v)\,dv \right\|_{H^{1/4}(\mathbb{R}\times\mathbb{R}^3)} \le c\left(\|f_\delta^n\|_2^2 + \|g_{1,\delta}^n\|_2^2 + \|g_{2,\delta}^n\|_2^2 \right)^{1/2} \tag{7.42}$$

where these L^2 norms are taken over $\mathbb{R}\times\mathbb{R}^3\times B_R$. By compactness, there exists a subsequence $\{n_k\}$ and a function $h_\delta \in L^2((0,T)\times B_S)$ such that

$$\int f_\delta^{n_k}(\cdot,\cdot,v)\psi(v)\,dv \to h_\delta \quad \text{in} \quad L^2((0,T)\times B_S). \tag{7.43}$$

Now diagonalize (by taking rational δ's) to obtain a subsequence independent of δ. Call it f_δ^n again. By weak convergence,

$$\int f_\delta^n(\cdot,\cdot,v)\psi(v)\,dv \rightharpoonup \int \zeta_\delta(\cdot)f(\cdot,\cdot,v)\psi(v)\,dv \quad \text{in } L^2((0,T)\times B_S). \tag{7.44}$$

Therefore

$$h_\delta = \int \zeta_\delta(\cdot)f(\cdot,\cdot,v)\psi(v)\,dv \quad \text{a.e. on} \quad [0,T]\times B_S. \tag{7.45}$$

In particular,

$$\int f_\delta^n(\cdot,\cdot,v)\psi(v)\,dv \to \int \zeta_\delta(\cdot)f(\cdot,\cdot,v)\psi(v)\,dv \quad \text{in } L^1((0,T)\times B_S). \tag{7.46}$$

Hence

$$\begin{aligned}
&\int_0^T \int_{B_S} \left| \int f^n(t,x,v)\psi(v)\,dv - \int f(t,x,v)\psi(v)\,dv \right| dx\,dt \\
&= \int_0^\delta \int_{B_S} \left| \int f^n\psi\,dv - \int f\psi\,dv \right| dx\,dt \\
&\quad + \int_\delta^T \int_{B_S} \left| \int \zeta_\delta(t) f^n\psi\,dv - \int \zeta_\delta f\psi\,dv \right| dx\,dt \\
&\le \|\psi\|_\infty \sup_n \int_0^\delta \int_{B_S}\int_{B_R} f^n\,dv\,dx\,dt + \|\psi\|_\infty \int_0^\delta \int_{B_S}\int_{B_R} f\,dv\,dx\,dt \\
&\quad + \left\| \int f_\delta^n\psi\,dv - \int \zeta_\delta f\psi\,dv \right\|_{L^1((0,T)\times B_S)}.
\end{aligned} \tag{7.47}$$

The last term tends to 0 as $n\to\infty$ by (7.46). For the second term we have

$$0 \le f \in L^2((0,T)\times B_S\times B_R) \tag{7.48}$$

because the norm is weakly lower semi-continuous. Hence

$$0 \le f \in L^1((0,T)\times B_S\times B_R) \tag{7.49}$$

and thus the second term can be made arbitrarily small for δ small by absolute continuity. The same argument applies to the first term in (7.47).

Step 2: Assume only $\psi \in \mathcal{D}(\mathbb{R}^3)$. Let

$$\beta_\delta(\tau) = \frac{\tau}{1+\delta\tau} \quad \text{for} \quad \delta > 0,\ \tau \geq 0.$$

Thus $\beta_\delta(0) = 0$, and the triple $(\beta_\delta(f^n),\ E^n,\ B^n)$ satisfy (MVM). Moreover, we clearly have $\beta_\delta(f^n) \leq \delta^{-1}$ so $\beta_\delta(f^n)$ is bounded in $L^\infty((0,\infty) \times \mathbb{R}^3 \times \mathbb{R}^3)$. As before, there exists a subsequence such that

$$\beta_\delta(f^n) \rightharpoonup f_\delta \quad \text{in } L^2((0,T) \times \mathbb{R}^6) \ \ (\text{because } \beta_\delta(f^n) \leq f^n) \tag{7.50}$$

and for fixed T, S,

$$\int \beta_\delta(f^n(\cdot,\cdot,v))\psi(v)\,dv \to \int f_\delta(\cdot,\cdot,v)\psi(v)\,dv \tag{7.51}$$

(through a subsequence) in $L^1((0,T) \times B_S)$. The subsequence can be chosen independently of δ via diagonalization.

Now since $\beta_\delta(f^n) \leq f^n$, we have for every bounded measurable $A \subset (0,\infty) \times \mathbb{R}^6$,

$$\int_A f_\delta\,dv\,dx\,dt = \lim_{n\to\infty} \int_A \beta_\delta(f^n)\,dv\,dx\,dt \leq \lim_{n\to\infty} \int_A f^n\,dv\,dx\,dt = \int_A f\,dv\,dx\,dt.$$

Thus $f_\delta \leq f$ a.e. Therefore

$$\begin{aligned} \int_{(0,T)\times B_S \times B_R} |f - f_\delta|\,dv\,dx\,dt &= \int_{(0,T)\times B_S \times B_R} (f - f_\delta)\,dv\,dx\,dt \\ &= \lim_{n\to\infty} \int_{(0,T)\times B_S \times B_R} (f^n - \beta_\delta(f^n))\,dv\,dx\,dt \\ &\leq \sup_n \int_{(0,T)\times B_S \times B_R} (f^n - \beta_\delta(f^n))\,dv\,dx\,dt. \end{aligned} \tag{7.52}$$

For R such that $\operatorname{supp}\psi \subset B_R$,

$$\begin{aligned} &\int_0^T \int_{B_S} \left| \int f^n\psi\,dv - \int f\psi\,dv \right| dx\,dt \\ &\leq \int_0^T \int_{B_S} \left| \int \beta_\delta(f^n)\psi\,dv - \int f_\delta\psi\,dv \right| dx\,dt \\ &\quad + \int_0^T \int_{B_S} \left| \int f^n\psi\,dv - \int \beta_\delta(f^n)\psi\,dv \right| dx\,dt \\ &\quad + \int_0^T \int_{B_S} \left| \int f_\delta\psi\,dv - \int f\psi\,dv \right| dx\,dt \\ &\leq \int_0^T \int_{B_S} \left| \int \beta_\delta(f^n)\psi\,dv - \int f_\delta\psi\,dv \right| dx\,dt \\ &\quad + 2\|\psi\|_\infty \sup_n \int_0^T \int_{B_S} \int_{B_R} (f^n - \beta_\delta(f^n))\,dv\,dx\,dt. \end{aligned} \tag{7.53}$$

By the above L^1-convergence, the first term tends to 0 as $n \to \infty$. For the second term we compute

$$0 \leq f^n - \beta_\delta(f^n) = f^n - \frac{f^n}{1+\delta f^n} = \frac{\delta (f^n)^2}{1+\delta f^n} \leq \delta (f^n)^2. \tag{7.54}$$

Thus the second term is less than

$$2\|\psi\|_\infty \delta T c_2.$$

This completes Step 2.

Step 3: Let $\psi \in C_0^0(\mathbb{R}^3)$. Take $\psi_m \in \mathcal{D}(\mathbb{R}^3)$ with $\|\psi_m - \psi\|_2 \to 0$; T, S fixed. By Step 2, there is a subsequence such that, for each m,

$$\int f^n \psi_m \, dv \overset{n\to\infty}{\longrightarrow} \int f \psi_m \, dv \quad \text{in } L^1((0,T) \times B_S). \tag{7.55}$$

Therefore

$$\begin{aligned}
&\int_0^T \int_{B_S} \left| \int f^n \psi \, dv - \int f \psi \, dv \right| dx\, dt \\
&\leq \int_0^T \int_{B_S} \int f^n |\psi - \psi_m| \, dv\, dx\, dt + \int_0^T \int_{B_S} \left| \int f^n \psi_m \, dv - \int f \psi_m \, dv \right| dx\, dt \\
&\quad + \int_0^T \int_{B_S} \int f |\psi - \psi_m| \, dv\, dx\, dt \\
&\leq \|f^n\|_{L^2((0,T)\times B_S \times \mathbb{R}^3)} \cdot \|\psi - \psi_m\|_2 \cdot c_{T,S} \\
&\quad + \int_0^T \int_{B_S} \left| \int f^n \psi_m \, dv - \int f \psi_m \, dv \right| dx\, dt \\
&\quad + \|f\|_{L^2((0,T)\times B_S \times \mathbb{R}^3)} \cdot \|\psi - \psi_m\|_2 \cdot c_{T,S}.
\end{aligned}$$

Choose m first so that the first and third terms are arbitrarily small. Then take n large to make the second term small.

Step 4: Let ψ be as in the Theorem, $\psi(v) = o(|v|^2)$, $|v| \to \infty$. Take $\zeta_M \in \mathcal{D}(\mathbb{R}^3)$, $\zeta_M \equiv 1$ on $\overline{B}_M$, $0 \leq \zeta_M \leq 1$, for $M \in \mathbb{N}$. Apply Step 3 to $\zeta_M \psi$. By diagonalization, for fixed $T > 0$, $S > 0$, there is a subsequence f^n such that for all $M \in \mathbb{N}$,

$$\int f^n \zeta_M \psi \, dv \to \int f \zeta_M \psi \, dv \quad \text{as} \quad n \to \infty$$

in $L^1((0,T) \times B_S)$. Write

$$\begin{aligned}
&\int_0^T \int_{B_S} \left| \int f^n \psi \, dv - \int f \psi \, dv \right| dx\, dt \\
&\leq \sup_n \int_0^T \int_{B_S} \int f^n (1-\zeta_M) |\psi| \, dv\, dx\, dt \\
&\quad + \int_0^T \int_{B_S} \left| \int f^n \zeta_M \psi \, dv - \int f \zeta_M \psi \, dv \right| dx\, dt \\
&\quad + \int_0^T \int_{B_S} \int f (1-\zeta_M) |\psi| \, dv\, dx\, dt.
\end{aligned} \tag{7.56}$$

For fixed M, the second term $\to 0$ by the strong L^1 convergence above. The third term $\to 0$ as $M \to \infty$ by dominated convergence. For the first term we have

$$\begin{aligned} (7.57) \qquad & \sup_n \int_0^T \int_{B_S} f^n(1-\zeta_M)|\psi|\,dv\,dx\,dt \\ & \le \sup_{|v|\ge M} \frac{|\psi(v)|}{|v|^2} \cdot \sup_n \int_0^T \int_{B_S} \int_{|v|>M} f^n |v|^2\,dv\,dx\,dt \\ & \le \sup_{|v|\ge M} \frac{|\psi(v)|}{|v|^2} \cdot c_3 T \to 0 \quad \text{as } M \to \infty. \end{aligned}$$

This completes the proof of Theorem 7.3.1.

COROLLARY *Assume in addition that for all $T > 0$, $P > 0$,*

$$(7.58) \qquad \{f^n(t,\cdot,\cdot)^2 : n \in \mathbb{N},\ t \in [0,T]\} \quad \textit{is relatively weakly compact in } L^1(B_P \times B_P).$$

Then for all $\psi \in \mathcal{D}(\mathbb{R}^3)$, there exists a subsequence such that

$$(7.59) \qquad \int f^n(\cdot,\cdot,v)\psi(v)\,dv \to \int f(\cdot,\cdot,v)\psi(v)\,dv$$

strongly in $L^2((0,T) \times B_S)$.

Remark: Given this we can pass to the limit in $\mathcal{D}'$ in the approximate equations for (f^n, E^n, B^n) and thus there exists a weak global solution.

Proof. We know that $\int f^n(\cdot,\cdot,v)\psi(v)\,dv \to \int f(\cdot,\cdot,v)\psi(v)\,dv$ in $L^1((0,T)\times B_S)$ through a subsequence. Passing to yet another subsequence, we can assume that this convergence also holds a.e. on $(0,T) \times B_S$. Now we claim that $\{(\int f^n(\cdot,\cdot,v)\psi(v)\,dv)^2\}$ is relatively weakly compact in $L^1((0,T) \times B_S)$. We use the Dunford–Pettis Theorem for this (cf. [5, p. 292]).

For the proof of the claim, we need to show that

(i) $\{(\int f^n\psi\,dv)^2\}$ is bounded in $L^1((0,T) \times B_S)$;

(ii) for all $\epsilon > 0$, there exists $\delta > 0$ such that for all $A \subset (0,T) \times B_S$ with $|A| \le \delta$, and for all n,

$$(7.60) \qquad \iint_A \left(\int f^n\psi\,dv\right)^2 dx\,dt \le \epsilon.$$

(i) follows from the hypotheses of the Theorem and the Schwarz inequality. For (ii): take $P = \max(R, S)$ (where $\operatorname{supp}\psi \subset B_R$). Write $A_t = \{x \in B_S : (t,x) \in A\}$. Let A be measurable with $|A| \le \delta^2$. Then

$$(7.61) \qquad \iint_A \left(\int f^n\psi\,dv\right)^2 dx\,dt = \int_0^T \int_{B_S} \chi_{A_t}(x) \left(\int_{B_R} f^n\psi\,dv\right)^2 dx\,dt$$

$$
\begin{aligned}
&= \int_{\{t\in(0,T):|A_t|>\delta\}} \int_{B_S} \chi_{A_t}(x) \left(\int_{B_R} f^n \psi \, dv \right)^2 dx\, dt \\
&\quad + \int_{\{t\in(0,T):|A_t|\le\delta\}} \int_{B_S} \chi_{A_t}(x) \left(\int_{B_R} f^n \psi \, dv \right)^2 dx\, dt \\
&\equiv L_1 + L_2.
\end{aligned}
$$

Now we assert that $|\{t \in (0,T) : |A_t| > \delta\}| \le \delta$. For if not, we get

$$
|A| = \int_0^T |A_t|\, dt \ge \int_{\{t\in(0,T):|A_t|>\delta\}} |A_t|\, dt > \delta^2
$$

which is a contradiction to the size of the measure of A. Therefore

$$
\begin{aligned}
L_1 &\le \|\psi\|_2^2 \int_{\{t\in(0,T):|A_t|>\delta\}} \int_{B_S} \int_{B_R} |f^n|^2 \, dv\, dx\, dt \\
&\le c_2 \delta \|\psi\|_2^2.
\end{aligned} \tag{7.62}
$$

Next,

$$
L_2 \le \|\psi\|_2^2 \int_{\{t\in(0,T):|A_t|\le\delta\}} \int_{B_S} \int_{B_R} |f^n|^2 \, dv\, dx\, dt. \tag{7.63}
$$

This integral can be made arbitrarily small in view of $|A_t \times B_R| \le \delta |B_R|$ and the hypothesis of weak L^1 compactness. (The Dunford–Pettis theorem gives a characterization of weak compactness in L^1 via a necessary and sufficient condition; thus the remaining integral is taken only over a set of small measure, and so the integral itself can be made arbitrarily small). This proves (ii) and the weak L^1-compactness.

Now, to finish the proof of the Corollary, we apply the Vitali Convergence Theorem. Because we are working on a bounded set, we need two ingredients. They are the pointwise convergence of a subsequence

$$
\int f^n(\cdot,\cdot,v)\psi(v)\, dv \to \int f(\cdot,\cdot,v)\psi(v)\, dv \quad \text{a.e. on} \quad (0,T)\times B_S
$$

(which follows from the already established L^1-convergence) and the estimate (7.60). Strong convergence of a subsequence in $L^2((0,T)\times B_S)$ now follows.

7.4. Completion of the Proof

Several details remain to be treated. These include the regularization of the initial data, the manner in which the initial data are assumed, and the satisfaction of the divergence–type constraints in the Maxwell equations. We will sketch some of these arguments; full details may be found in Theorem 4.1.1 in [14] and in [3].

We assume that the compatible initial data satisfies the following:

$$
\iint (1+|v|^2) f_0 \, dv\, dx < \infty, \quad f_0 \ge 0 \text{ a.e.}, \quad f_0 \in L^2(\mathbb{R}^6), \quad E_0, B_0 \in L^2(\mathbb{R}^3).
$$

We find a sequence $f_0^n \geq 0$ such that

$$f_0^n \to f_0 \quad \text{in} \quad L^2(\mathbb{R}^6), \quad f_0^n \in \mathcal{D} \quad \text{and} \quad \iint (1+|v|^2)|f_0^n - f_0|\, dv\, dx \to 0.$$

Similarly, we approximate in $L^2(\mathbb{R}^3)$-norm the data for E, B by test functions E_0^n, B_0^n. Then for every n and every $T>0$, there exists P such that for all $t\in[0,T]$, $\operatorname{supp} f^n(t,\cdot,\cdot) \subset B_P \times B_P$. We have

$$\iint |f^n|^2\, dv\, dx = \iint (f_0^n)^2\, dv\, dx \leq c_2; \tag{7.64}$$
$$\iint f^n\, dv\, dx = \iint f_0^n\, dv\, dx \leq c_1.$$

From the approximation of f_0 in L^2 by f_0^n, we have the existence of a non-negative function $\beta \in C^\infty(\mathbb{R})$ which satisfies $\beta(0)=0$, $\lim_{t\to\infty} t^{-2}\beta(t) = \infty$, and

$$\sup_n \iint \beta(f_0^n)\, dv\, dx < \infty. \tag{7.65}$$

Hence also

$$\iint \beta(f^n)\, dv\, dx \leq \text{const.} \equiv S_1.$$

For the existence of such a function β, see [14], Lemma 4.3.6 where a proof of G. Schlüchtermann is presented.

By the Dunford–Pettis Theorem, we need to show that for each $\epsilon > 0$, there exists $\delta > 0$ such that for all measurable $A \subset B_P \times B_P$ with $|A| \leq \delta$, and for all n and for all $t \in [0,T]$ we have $\iint_A |f^n|^2\, dv\, dx \leq \epsilon$.

Let $\epsilon > 0$. Choose $\sigma > 0$ such that $\frac{\beta(\tau)}{\tau^2} \geq \frac{2S_1}{\epsilon}$ for all $\tau \geq \sigma$. Let $\delta \leq \frac{\epsilon}{2\sigma^2}$; let $A \subset B_P \times B_P$ have $|A| \leq \delta$. Then

$$\begin{aligned}\iint_A |f^n|^2\, dv\, dx &= \left(\int_{A\cap\{f^n(t,\cdot,\cdot)\leq\sigma\}} + \int_{A\cap\{f^n(t,\cdot,\cdot)>\sigma\}}\right)|f^n|^2\, dv\, dx \\ &\leq \sigma^2 \int_{A\cap\{f^n\leq\sigma\}} dv\, dx + \frac{\epsilon}{2S_1}\int_{A\cap\{f^n>\sigma\}} \beta(f^n)\, dv\, dx \\ &\leq \sigma^2|A| + \frac{\epsilon}{2S_1}\cdot S_1 \leq \epsilon\end{aligned} \tag{7.66}$$

as desired.

Thus the major theorem is established: the existence of a global weak solution (f,E,B) to the initial value problem for the Vlasov–Maxwell System ((7.1) with $j^\epsilon = j$). As for the regularity of solutions, one gets by this method

$$f \in L^\infty([0,\infty);\ L^1(\mathbb{R}^6)\cap L^2(\mathbb{R}^6))$$
$$E, B \in L^\infty([0,\infty);\ L^2(\mathbb{R}^3))\ .$$

Moreover, for suitably chosen representatives we have

$$\begin{aligned} &f \in C([0,\infty);\ W^{-s,2}(B^6_P)) && \text{for every } s>0,\ P>0,\\ &E, B \in C([0,\infty);\ W^{-s,2}(B^3_P)) && \text{for every } s>0, P>0 \end{aligned}$$

and the initial values are assumed in this sense. These continuity properties follow from applications of the Arzela Ascoli theorem and the Rellich compactness theorem. Furthermore, the following quantities are dominated at any time $t>0$ by their initial values: the $L^2(\mathbb{R}^6)$–norm of f, and the total energy (the left–hand side of the second expression in (7.2) with the superscripts of n dropped), while the $L^1(\mathbb{R}^6)$–norm of f is invariant.

In the sense of distributions one shows that $\partial_t\rho + \nabla_x \cdot j = 0$ and then, using this and the Maxwell equations themselves, one shows (again in the sense of distributions) that

$$\partial_t(\nabla_x \cdot E - \rho) = 0, \quad \partial_t(\nabla_x \cdot B) = 0.$$

In view of the constraints imposed on the initial values, we see that the time independent divergence–type constraints in the Maxwell equations then hold.

The relativistic version of this weak global solvability is also treated in [14].

For the approach developed in this chapter, no higher regularity nor uniqueness is known at present. An interesting feature is that there is essentially no difference between the nonrelativistic and relativistic problems in this method. Yet for smooth solvability, we have seen drastic differences between the two problems in Chapters 5 and 6.

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Chapter 8

CONVERGENCE OF A PARTICLE METHOD FOR THE VLASOV–MAXWELL SYSTEM

8.1. Introduction

As we have seen, the Vlasov–Maxwell system models the time evolution of a collisionless plasma. Here we study a particle simulation for the relativistic Vlasov–Maxwell system. In one space dimension (and only one momentum dimension) Maxwell's equations degenerate to Poisson's equation. Particle simulation of this case has been analyzed by Cottet and Raviart [2], [3] (see also [18], [19]). Particle methods have also been analyzed for higher-dimensional problems [6] where Maxwell's equations are replaced by Poisson's equation. A particle method for a three-dimensional symmetric Vlasov-Poisson system is treated in [16]. In [13], [14] convergence results in a measure theoretic sense are presented, while a general introduction to particle methods may be found in [15].

In order to retain the hyperbolic structure of the problem we consider the "one and one–half" dimensional model, in which there is only one spatial variable but two momentum variables. We consider the relativistic version of this model; hence from [7] we know the global existence of smooth solutions and, moreover, a uniform (in space and time) bound on the electromagnetic fields. The analysis presented here is taken from [8] and appears to be the first to obtain a convergence result for a particle method where the coupling in the underlying problem involves the Maxwell equations. This algorithm has been implemented in [9]. Particle methods are good choices for several reasons. Among these are the natural use of finitely many particles to model a continuum, and the preservation of the positivity of the phase–space densities. Related algorithms appear in [10], [11].

Thus we consider the "one and one–half dimensional" Vlasov–Maxwell system with one species of charge in a neutralizing background density. We seek $f(t,x,p)$, $E(t,x)$, $B(t,x)$ (where $t \geq 0$; x, f, B are scalar; $p = (p_1, p_2)$ and $E = (E_1, E_2)$) such that

$$\partial_t f + v_1(p)\partial_x f + q(E + BMv(p)) \cdot \nabla_p f = 0,$$

$$\text{(RVM)} \qquad \partial_t E_1 = -4\pi j_1, \qquad \partial_x E_1 = 4\pi\rho,$$

$$\begin{aligned}\partial_t E_2 &= -\partial_x B - 4\pi j_2,\\ \partial_t B &= -\partial_x E_2\end{aligned}$$

with

$$\begin{aligned}&f(0,x,p) = f^0(x,p) \geq 0,\\ &E_2(0,x) = E_2^0(x), \qquad B(0,x) = B^0(x),\end{aligned}$$

and

$$E_1(0,x) = 4\pi \int_{-\infty}^{x} \rho(0,y)\,dy = E_1^0(x).$$

Here

$$v(p) = (m^2 + |p|^2)^{-1/2} p, \qquad M = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

$$\begin{aligned}j(t,x) &= q \int f(t,x,p) v(p)\,dp,\\ \rho(t,x) &= q \int f(t,x,p)\,dp - \nu(x)\end{aligned}$$

where the given neutralizing background density $\nu(x)$ has compact support and is chosen so that

$$\int \rho(0,x)\,dx = 0.$$

When $m = 1$, $v(p) = \hat{p}$ in our notation in previous chapters. We make this change in order to simplify the notation for the algorithm below. We assume that all data functions are smooth and compactly supported. Note that the initial condition for E_1 is prescribed by $f(0,x,v)$. In fact, since

$$\partial_x E_1 = 4\pi\rho$$

we have

$$E_1(t,x) = 4\pi \int_{-\infty}^{x} \rho(t,y)\,dy + \text{const.}$$

But by the initial condition for E_1 we have

$$E_1(t,x) = \mathcal{E}_1[\rho]|_{(t,x)}, \tag{8.1}$$

where we define

$$\mathcal{E}_1[\sigma]|_{(t,x)} = 4\pi \int_{-\infty}^{x} \sigma(t,y)\,dy. \tag{8.2}$$

Note also that

$$\begin{aligned}\partial_t \left(4\pi \int_{-\infty}^{x} \rho(t,y)\,dy\right) &= \partial_t \left(4\pi \int_{-\infty}^{x} \left(q \int f(t,y,p)\,dp - \nu(y)\right) dy\right)\\ &= -4\pi q \int_{-\infty}^{x} \int v_1(p) \partial_x f(t,y,p)\,dp\,dy\\ &= -4\pi q \int v_1(p) f(t,x,p)\,dp\\ &= -4\pi j_1(t,x),\end{aligned}$$

so the two equations

$$\partial_t E_1 = -4\pi j_1 \quad \text{and} \quad \partial_x E_1 = 4\pi\rho$$

are consistent.

Note that the speed of light has been normalized to unity. Henceforth we will also take the charge q and the rest mass m of each charged particle to be unity.

In order to get a representation for the fields, we write

$$\partial_t(E_2 + B) + \partial_x(E_2 + B) = -4\pi j_2,$$

so for any $h \geq 0$

$$(E_2 + B)|_{(t+h,x)} = (E_2 + B)|_{(t,x-h)} - 4\pi \int_t^{t+h} j_2(\tau, x - [t + h - \tau])\, d\tau. \tag{8.3}$$

Similarly,

$$(E_2 - B)|_{(t+h,x)} = (E_2 - B)|_{(t,x+h)} - 4\pi \int_t^{t+h} j_2(\tau, x + [t + h - \tau])\, d\tau, \tag{8.4}$$

and hence

$$\begin{aligned} E_2(t+h,x) = {} & \frac{1}{2}[E_2(t,x-h) + E_2(t,x+h) + B(t,x-h) - B(t,x+h)] \\ & - 2\pi \int_t^{t+h} (j_2(\tau, x - t - h + \tau) + j_2(\tau, x + t + h - \tau))\, d\tau \end{aligned} \tag{8.5}$$

and

$$\begin{aligned} B(t+h,x) = {} & \frac{1}{2}[E_2(t,x-h) - E_2(t,x+h) + B(t,x-h) + B(t,x+h)] \\ & - 2\pi \int_t^{t+h} (j_2(\tau, x - t - h + \tau) - j_2(\tau, x + t + h - \tau))\, d\tau. \end{aligned} \tag{8.6}$$

Define the characteristics $\mathcal{X}(s,t,x,p)$ and $\mathcal{P}(s,t,x,p)$ by

$$\begin{aligned} &\frac{d}{ds}\mathcal{X} = v_1(\mathcal{P}), && \mathcal{X}(t,t,x,p) = x, \\ &\frac{d}{ds}\mathcal{P} = E(s,\mathcal{X}) + B(s,\mathcal{X})Mv(\mathcal{P}), && \mathcal{P}(t,t,x,p) = p. \end{aligned} \tag{8.7}$$

Now $f(t,x,p)$ is nonnegative and uniformly bounded since for $s \geq 0$ we have

$$\begin{aligned} f(t,x,p) &= f(s, \mathcal{X}(s,t,x,p), \mathcal{P}(s,t,x,p)) \\ &= f^0(\mathcal{X}(0,t,x,p), \mathcal{P}(0,t,x,p)). \end{aligned} \tag{8.8}$$

Also, as in Chapter 5 (see also [7]), the map

$$(x,p) \mapsto (\mathcal{X}(s,t,x,p), \mathcal{P}(s,t,x,p))$$

is measure preserving.

8.2. The Particle Simulation

Let Δx, Δp, ε be positive. We will denote $\Delta = \max\{\Delta x, \Delta p\}$. Define

$$C^\alpha = \{(x,p) : \alpha_1 \Delta x \le x < (\alpha_1+1)\Delta x,\ \alpha_2 \Delta p \le p_1 < (\alpha_2+1)\Delta p,\ \alpha_3 \Delta p \le p_2 < (\alpha_3+1)\Delta p\}$$

and

$$C^\alpha = \left(\left(\alpha_1 + \frac{1}{2}\right)\Delta x, \left(\alpha_2 + \frac{1}{2}\right)\Delta p, \left(\alpha_3 + \frac{1}{2}\right)\Delta p\right).$$

Let

$$q^\alpha = f^0(C^\alpha)\Delta x(\Delta p)^2;$$

this will be the charge of a particle whose initial state will be C^α. Let

$$\mathcal{A} = \{\alpha \in \mathbb{Z}^3 : q^\alpha \neq 0\}.$$

This set is finite by hypothesis.

Define S, δ_ε, and θ_ε by

$$S(x) = \begin{cases} 1 - |x| & \text{if } |x| \le 1, \\ 0 & \text{if } |x| > 1, \end{cases}$$

$$\delta_\varepsilon(x) = \varepsilon^{-1} S(\varepsilon^{-1} x),$$

and

$$\theta_\varepsilon(x) = \int_{-\infty}^{x} \delta_\varepsilon(y)\, dy.$$

δ_ε will give the "shape" of each particle.

We also define a uniform grid on $[0,\infty) \times \mathbb{R}$:

$$\begin{aligned} t^n &= n\varepsilon \quad \text{for } n = 0, \tfrac{1}{2}, 1, \tfrac{3}{2}, \cdots, \\ x^k &= k\varepsilon \quad \text{for } k \in \mathbb{Z}. \end{aligned}$$

Note that the time step and the spatial step in this grid are both ε, which is (so far) unrelated to Δx or Δp. Quantities pertaining to this grid are indexed with Roman letters, e.g., t^n and x^k. Quantities pertaining to the phase space grid are indexed with Greek letters, e.g., C^α.

To start the simulation define

$$\begin{aligned} \tilde{E}(0,x^k) &= E^0(x^k), \\ \tilde{B}(0,x^k) &= B^0(x^k), \\ \tilde{\mathcal{X}}^\alpha(0) &= \left(\alpha_1 + \frac{1}{2}\right)\Delta x, \\ \tilde{\mathcal{P}}^\alpha(t^{1/2}) &= \mathcal{P}(t^{1/2}, 0, C^\alpha), \end{aligned} \tag{8.9}$$

for all $k \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}^3$. We note that $\mathcal{P}(t^{1/2}, 0, C^\alpha)$ would not be known in practice, but we consider this definition of $\tilde{\mathcal{P}}^\alpha(t^{1/2})$ for simplicity. A comment on the inclusion of these errors appears in the "Summing the Errors" section below.

To define the simulation iteratively assume that for some $n \in \{0, 1, 2, \cdots\}$

$$\tilde{E}(t^n, x^k),\ \tilde{B}(t^n, x^k),\ \tilde{\mathcal{X}}^\alpha(t^n),\ \tilde{\mathcal{P}}^\alpha(t^{n+1/2})$$

are known approximations of

$$E(t^n, x^k),\ B(t^n, x^k),\ \mathcal{X}(t^n, 0, C^\alpha),\ \mathcal{P}(t^{n+1/2}, 0, C^\alpha)$$

for all $k \in \mathbb{Z}$ and $\alpha \in \mathbb{Z}^3$.

First define

$$\tilde{\mathcal{X}}^\alpha(t) = \tilde{\mathcal{X}}^\alpha(t^n) + (t - t^n) v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})) \tag{8.10}$$

for all $t \in [t^n, t^{n+1}]$. Next define

$$\tilde{\rho}(t^{n+1}, x) = \sum_\alpha q^\alpha \delta_\varepsilon(x - \tilde{\mathcal{X}}^\alpha(t^{n+1})) - \nu(x) \tag{8.11}$$

for all $x \in \mathbb{R}$, and

$$\tilde{E}_1(t^{n+1}, x^k) = \mathcal{E}_1[\tilde{\rho}](t^{n+1}, x^k)$$

for all $k \in \mathbb{Z}$. Note that $\mathcal{E}_1$ is the exact solution operator. We may compute this explicitly:

$$\begin{aligned} \tilde{E}_1(t^{n+1}, x^k) &= 4\pi \int_{-\infty}^{x^k} \tilde{\rho}(t^{n+1}, y)\, dy \\ &= 4\pi \sum_\alpha q^\alpha \int_{-\infty}^{x^k} \delta_\varepsilon(y - \tilde{\mathcal{X}}^\alpha(t^{n+1}))\, dy - 4\pi \int_{-\infty}^{x^k} \nu(y)\, dy \\ &= 4\pi \sum_\alpha q^\alpha \theta_\varepsilon(x^k - \tilde{\mathcal{X}}^\alpha(t^{n+1})) - 4\pi \int_{-\infty}^{x^k} \nu(y)\, dy. \end{aligned} \tag{8.12}$$

Next for all $x \in \mathbb{R}$ and $t \in [t^n, t^{n+1})$ define

$$\tilde{j}_2(t, x) = \sum_\alpha q^\alpha \delta_\varepsilon \left(x - \tilde{\mathcal{X}}^\alpha(t)\right) v_2 \left(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})\right), \tag{8.13}$$

and then (for all $k \in \mathbb{Z}$) define

$$\begin{aligned} \tilde{E}_2(t^{n+1}, x^k) = \frac{1}{2} &\left[\tilde{E}_2(t^n, x^{k-1}) + \tilde{E}_2(t^n, x^{k+1}) + \tilde{B}(t^n, x^{k-1}) - \tilde{B}(t^n, x^{k+1})\right] \\ &- 2\pi \int_{t^n}^{t^{n+1}} \left[\tilde{j}_2(\tau, x^{k-1} - t^n + \tau) + \tilde{j}_2(\tau, x^{k+1} + t^n - \tau)\right] d\tau \end{aligned} \tag{8.14}$$

and

$$\tilde{B}(t^{n+1}, x^k) = \frac{1}{2}\left[\tilde{E}_2(t^n, x^{k-1}) - \tilde{E}_2(t^n, x^{k+1}) + \tilde{B}(t^n, x^{k-1}) + \tilde{B}(t^n, x^{k+1})\right]$$

$$- 2\pi \int_{t^n}^{t^{n+1}} \left[\tilde{j}_2(\tau, x^{k-1} - t^n + \tau) - \tilde{j}_2(\tau, x^{k+1} + t^n - \tau)\right] d\tau. \tag{8.15}$$

Again note that the exact field operators are used. Due to the special structure of $\tilde{j}_2$ (constant velocities) we may compute $\tilde{E}_2$ and $\tilde{B}$ explicitly. We need the value of

$$\int_{t^n}^{t^{n+1}} \tilde{j}_2(\tau, x^{k-1} - t^n + \tau)\, d\tau$$
$$= \sum_\alpha q^\alpha v_2(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})) \int_{t^n}^{t^{n+1}} \delta_\varepsilon(x^{k-1} - t^n + \tau - \tilde{\mathcal{X}}^\alpha(\tau))\, d\tau.$$

Let

$$\begin{aligned} z(\tau) &= x^{k-1} - t^n + \tau - \tilde{\mathcal{X}}^\alpha(\tau) \\ &= x^{k-1} - t^n + \tau - \tilde{\mathcal{X}}^\alpha(t^n) - v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))(\tau - t^n). \end{aligned}$$

Then $z'(\tau) = 1 - v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))$ so that

$$\int_{t^n}^{t^{n+1}} \tilde{j}_2(\tau, x^{k-1} - t^n + \tau)\, d\tau$$
$$= \sum_\alpha q^\alpha v_2\left(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})\right)\left[1 - v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))\right]^{-1} \int_{t^n}^{t^{n+1}} \delta_\varepsilon(z(\tau)) z'(\tau)\, d\tau$$
$$= \sum_\alpha q^\alpha v_2\left(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})\right) \frac{\theta_\varepsilon(z(t^{n+1})) - \theta_\varepsilon(z(t^n))}{1 - v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))}$$
$$= \sum_\alpha q^\alpha v_2\left(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})\right)\left[\frac{\theta_\varepsilon(x^k - \tilde{\mathcal{X}}^\alpha(t^{n+1})) - \theta_\varepsilon(x^{k-1} - \tilde{\mathcal{X}}^\alpha(t^n))}{1 - v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))}\right].$$

Similarly,

$$\int_{t^n}^{t^{n+1}} \tilde{j}_2(\tau, x^{k+1} + t^n - \tau)\, d\tau$$
$$= \sum_\alpha q^\alpha v_2\left(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})\right) \int_{t^n}^{t^{n+1}} \delta_\varepsilon(x^{k+1} + t^n - \tau - \tilde{\mathcal{X}}^\alpha(\tau))\, d\tau$$
$$= \sum_\alpha q^\alpha v_2\left(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})\right)\left(1 + v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))\right)^{-1}$$
$$\cdot \left(\theta_\varepsilon(x^{k+1} - \tilde{\mathcal{X}}^\alpha(t^n)) - \theta_\varepsilon(x^k - \tilde{\mathcal{X}}^\alpha(t^{n+1}))\right).$$

Now that $\tilde{E}(t^{n+1}, x)$ and $\tilde{B}(t^{n+1}, x)$ are defined for $x \in \{x_k : k \in \mathbb{Z}\}$, we define them for all $x \in \mathbb{R}$ by linear interpolation. Although definitions (8.14) and (8.15) could be used for values of $x \notin \{x_k : k \in \mathbb{Z}\}$, this would require much more computation. We also comment that the linear interpolation may be written as (for example)

$$B(t^n, x) = \sum_k B(t^n, x^k)\delta_\varepsilon(x - x^k)$$

which indicates that the method is momentum preserving as described in [1].

Finally, we define $\tilde{\mathcal{P}}^\alpha(t^{n+3/2})$ implicitly by

$$\begin{aligned} \tilde{\mathcal{P}}^\alpha(t^{n+3/2}) &= \tilde{\mathcal{P}}^\alpha(t^{n+1/2}) + \varepsilon\tilde{E}\left(t^{n+1}, \tilde{\mathcal{X}}^\alpha(t^{n+1})\right) \\ &\quad + \varepsilon\tilde{B}\left(t^{n+1}, \tilde{\mathcal{X}}^\alpha(t^{n+1})\right) M \cdot \frac{1}{2}\left[v(\tilde{\mathcal{P}}^\alpha(t^{n+1/2})) + v(\tilde{\mathcal{P}}^\alpha(t^{n+3/2}))\right]. \end{aligned} \tag{8.16}$$

Although (8.16) is a nonlinear system, it is always explicitly and uniquely solvable for $\tilde{\mathcal{P}}^\alpha(t^{n+3/2})$. We may obtain this solution by solving for $|\tilde{\mathcal{P}}^\alpha(t^{n+3/2})|$ first, and then by substituting this into (8.16) to obtain a pair of linear equations. Thus one step of the iteration has been completed.

We point out that the above scheme is *fully discrete*, but that difference schemes have not been used to solve Maxwell's equations. Rather, Maxwell's equations are solved *exactly* for approximate sources ($\tilde{\rho}$ and $\tilde{j}$), thus eliminating one corruption present in most discrete particle methods. This scheme may be implemented with an operation count on the order of $C(\Delta x)^{-1}(\Delta p)^{-2}$ per timestep. This is comparable to the operation count incurred when difference schemes are used to solve Maxwell's equations.

The main theorem is the following.

THEOREM 8.2.1 *Let smooth initial data for (RVM) be given as above and let ε, Δ, $\tilde{\mathcal{X}}^\alpha$, $\tilde{\mathcal{P}}^\alpha$, $\tilde{E}$, and $\tilde{B}$ be as above. Let $C_1 > 0$ be given; then there exists*

$$C : [0, \infty) \to [0, \infty)$$

such that for all ε and Δ such that

$$\Delta < C_1\varepsilon \quad \textit{and} \quad \varepsilon \sup_{x,t}|B(t,x)| < 1$$

we have

$$\begin{aligned} &\sup\left\{|\mathcal{X}(\tau, 0, C^\alpha) - \tilde{\mathcal{X}}^\alpha(\tau)| + |\mathcal{P}(\tau, 0, C^\alpha) - \tilde{\mathcal{P}}^\alpha(\tau)| : \alpha \in \mathcal{A},\ \tau \in [0, t]\right\} \\ &\qquad + \sup\left\{|E(\tau, x) - \tilde{E}(\tau, x)| + |B(\tau, x) - \tilde{B}(\tau, x)| : \tau \in [0, t],\ x \in \mathbb{R}\right\} \\ &\leq C(t)(\varepsilon + \varepsilon^{-1}\Delta^2) \end{aligned}$$

for all t in some time interval $[0, T_{\varepsilon,\Delta}]$. Furthermore,

$$\lim_{\substack{\varepsilon \to 0^+ \\ \Delta < C_1\varepsilon}} T_{\varepsilon,\Delta} = +\infty.$$

Comments

(1) From [7] it is known that B is uniformly bounded, so the "stability" condition

$$\varepsilon \sup |B| < 1$$

is attainable. Also, if the charge q and rest mass m were not unity, the "stability" condition would be

$$\varepsilon |q| m^{-1} \sup |B| < 1.$$

The bounds on E_2 and B result from an energy identity. Let

$$e(t,x) = \frac{1}{2}|E|^2 + \frac{1}{2}B^2 + 4\pi \int f \sqrt{1+|p|^2}\, dp.$$

Then

$$\partial_t e + \partial_x \left(E_2 B + 4\pi \int f p_1\, dp \right) = 0.$$

Given x and t we integrate this over

$$\{(\tau,y) : 0 \le \tau \le t \text{ and } |x-y| \le t-\tau\}$$

and apply Green's Theorem. There results (using $\frac{1}{2}|E|^2 + \frac{1}{2}B^2 \ge |E_2 B|$)

$$\begin{aligned}\int_{x-t}^{x+t} e(0,y)\, dy \ge 4\pi \int_0^t \int f(\tau, x-t+\tau, p) \left[\sqrt{1+|p|^2} - p_1\right] dp\, d\tau \\ + 4\pi \int_0^t \int f(\tau, x+t-\tau, p) \left[\sqrt{1+|p|^2} + p_1\right] dp\, d\tau.\end{aligned}$$

Now that $e(0,y)$ is integrable over $\mathbb{R}$ by assumption. A short computation shows that

$$\sqrt{1+|p|^2} \pm p_1 \ge |v(p_2)|$$

(this is done below), so

$$\begin{aligned}C &\ge 4\pi \int_0^t \int f(\tau, x-t+\tau, p) |v(p_2)|\, dp\, d\tau \\ &\quad + 4\pi \int_0^t \int f(\tau, x+t-\tau, p) |v(p_2)|\, dp\, d\tau \\ &\ge 4\pi \int_0^t [|j_2(\tau, x-t+\tau)| + |j_2(\tau, x+t-\tau)|]\, d\tau.\end{aligned}$$

Uniform bounds for E_2 and B now follow from (8.5) and (8.6) when we substitute $t=0$, $h=t$ in those equations.

(2) The function

$$\varepsilon \mapsto \varepsilon + \varepsilon^{-1}\Delta^2$$

is minimal when $\varepsilon = \Delta$, in which case

$$\varepsilon + \varepsilon^{-1}\Delta^2 = 2\Delta.$$

(3) The work [2] establishes the better estimate of

$$C(t)(\varepsilon^2 + \varepsilon^{-1}\Delta^2).$$

Here the extra power of ε is lost due to the fully discrete nature of the scheme.

For convenience we will write

$$\tilde{\mathcal{P}}^\alpha(t) = \tilde{\mathcal{P}}^\alpha(t^{n+1/2})$$

for $t \in [t^n, t^{n+1})$. Now we have

$$\tilde{\mathcal{X}}^\alpha(t) = \tilde{\mathcal{X}}^\alpha(0) + \int_0^t v_1(\tilde{\mathcal{P}}^\alpha(\tau))\, d\tau$$

for all $t \geq 0$. We will also write

$$S(t) = \sup\left\{ |\mathcal{P}(t,0,x,p)| : (x,p) \in \bigcup_{\alpha\in\mathcal{A}} C^\alpha \right\},$$
$$\tilde{S}(t) = \sup\{|\tilde{\mathcal{P}}^\alpha(t)| : \alpha \in \mathcal{A}\}.$$

It is shown in [7] that

$$S(t) \leq C(1+t)^2$$

but we are unable to obtain such an a priori bound for $\tilde{S}$. Thus convergence estimates will be made using $\tilde{S}$, then a posteriori, the convergence estimates will be used to bound $\tilde{S}$. Note that

$$\begin{aligned} 1 - |v_1(p)| &= 1 - \frac{|p_1|}{\sqrt{1+|p|^2}} \\ &= \frac{(\sqrt{1+|p|^2} - |p_1|)(\sqrt{1+|p|^2} + |p_1|)}{\sqrt{1+|p|^2}(\sqrt{1+|p|^2} + |p_1|)} \\ &\geq \frac{1+p_2^2}{2(1+|p|^2)} \\ &\geq \max\left\{ \frac{1}{2(1+|p|^2)}, \frac{|p_2|}{1+|p|^2} \right\} \end{aligned} \tag{8.17}$$

so for $(x,p) \in \operatorname{supp} f(t,\cdot,\cdot)$

$$\frac{1}{1-|v_1(p)|} \leq 2(1+S^2(t)) \leq C_T \tag{8.18}$$

and for $\alpha \in \mathbb{Z}^3$ such that $f^0(C^\alpha) \neq 0$,

$$\frac{1}{1-|v_1(\tilde{\mathcal{P}}^\alpha(t))|} \leq 2(1+\tilde{S}(t)^2). \tag{8.19}$$

We will also write

$$(8.20)\quad \tilde{\mathcal{E}}(t) = \max\Big\{ \sup\{|\mathcal{X}(\tau,0,C^\alpha) - \tilde{\mathcal{X}}^\alpha(\tau)| : \alpha \in \mathcal{A},\ 0 \le \tau \le t\}, \\ \sup\{|\mathcal{P}(\tau,0,C^\alpha) - \tilde{\mathcal{P}}^\alpha(\tau)| : \alpha \in \mathcal{A},\ 0 \le \tau < t\}\Big\}.$$

Constants C may depend on the Cauchy data but will be independent of t, x, p and of all numerical parameters $k, n, \alpha, \Delta x, \Delta p$ and ε. C_T denotes a similar constant, but one which will be used in estimates which need only hold for $0 \le t \le T$.

8.3. The Field Errors

Consider the errors in the fields at the gridpoint (t^n, x^k). From (8.3)

$$(E_2 + B)|_{(t^n,x^k)} = (E_2^0 + B^0)|_{x^k - t^n} - 4\pi \int_0^{t^n} j_2(\tau, x^k - t^n + \tau)\, d\tau.$$

Similarly, from (8.14) and (8.15)

$$\begin{aligned}(\tilde{E}_2 + \tilde{B})|_{(t^n,x^k)} &= (\tilde{E}_2 + \tilde{B})|_{(t^{n-1},x^{k-1})} - 4\pi \int_{t^{n-1}}^{t^n} \tilde{j}_2(\tau, x^k - t^n + \tau)\, d\tau \\ &= \cdots \\ &= (\tilde{E}_2 + \tilde{B})|_{(0,x^k - t^n)} - 4\pi \int_0^{t^n} \tilde{j}_2(\tau, x^k - t^n + \tau)\, d\tau.\end{aligned}$$

Now by (8.9)

$$[(E_2 + B) - (\tilde{E}_2 + \tilde{B})]|_{(t^n,x^k)} = -4\pi \int_0^{t^n} (j_2 - \tilde{j}_2)|_{(\tau, x^k - t^n + \tau)}\, d\tau.$$

Thus

$$(8.21)\qquad \frac{1}{4\pi}|(E_2 + B) - (\tilde{E}_2 + \tilde{B})||_{(t^n,x^k)} \le \mathrm{Moll} + \mathrm{Quad} + \mathrm{Sep}$$

where we define

$$(8.22)\ \mathrm{Moll} = \Big| \int_0^{t^n} \Big(j_2(\tau, x^k - t^n + \tau) - \int j_2(\tau, y)\delta_\varepsilon(x^k - t^n + \tau - y)\, dy \Big)\, d\tau \Big|,$$

$$(8.23)\ \mathrm{Quad} = \Big| \int_0^{t^n} \Big(\int j_2(\tau, y)\delta_\varepsilon(x^k - t^n + \tau - y)\, dy \\ - \sum_\alpha q^\alpha \delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha)) v_2(\mathcal{P}(\tau,0,C^\alpha)) \Big)\, d\tau \Big|,$$

and

$$(8.24)\quad \mathrm{Sep} = \Big|\sum_\alpha q^\alpha \int_0^{t^n} \Big(\delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha))v_2(\mathcal{P}(\tau,0,C^\alpha)) - \delta_\varepsilon(x^k - t^n + \tau - \tilde{\mathcal{X}}^\alpha(\tau))v_2(\tilde{\mathcal{P}}^\alpha(\tau))\Big)\,d\tau\Big|.$$

Note that by (8.8)

$$\begin{aligned}\int j_2(\tau,y)\delta_\varepsilon(x^k - t^n + \tau - y)\,dy &= \iint f(\tau,y,p)\delta_\varepsilon(x^k - t^n + \tau - y)v_2(p)\,dp\,dy\\ &= \iint f^0(\mathcal{X}(0,\tau,y,p),\mathcal{P}(0,\tau,y,p))\delta_\varepsilon(x^k - t^n + \tau - y)v_2(p)\,dp\,dy\\ &= \iint f^0(x,p)\delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,x,p))v_2(\mathcal{P}(\tau,0,x,p))\,dp\,dx,\end{aligned}$$

so

$$\begin{aligned}\mathrm{Quad} = \Big|\int_0^{t^n} \Big(&\iint f^0(x,p)\delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,x,p))v_2(\mathcal{P}(\tau,0,x,p))\,dp\,dx\\ &- \sum_\alpha f^0(C^\alpha)\Delta x(\Delta p)^2\delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha))v_2(\mathcal{P}(\tau,0,C^\alpha))\Big)\,d\tau\Big|. \qquad (8.25)\end{aligned}$$

Thus "Quad" is the error in the midpoint rule applied to the function

$$(x,p) \mapsto \int_0^{t^n} f^0(x,p)\delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,x,p))v_2(\mathcal{P}(\tau,0,x,p))\,d\tau.$$

Note that "Moll" and "Quad" involve only the exact solution, which is known to be smooth. Thus these two terms are the errors in standard approximations involving smooth functions, and may be estimated in a fairly standard way. However, the third term ("Sep") involves the separation of the exact and approximate characteristics, and is harder and more intimately linked with the specific problem. Thus we consider "Sep" first. We will need the following lemma.

LEMMA 8.3.1 *For $t \le T$ and $x \in \mathbb{R}$*

$$\begin{aligned}&\Big|\sum_\alpha q^\alpha \int_0^t (\delta_\varepsilon(x - t + \tau - \mathcal{X}(\tau,0,C^\alpha)) - \delta_\varepsilon(x - t + \tau - \tilde{\mathcal{X}}^\alpha(\tau)))\,d\tau\Big|\\ &\quad + \Big|\sum_\alpha q^\alpha \int_0^t (\delta_\varepsilon(x + t - \tau - \mathcal{X}(\tau,0,C^\alpha)) - \delta_\varepsilon(x + t - \tau - \tilde{\mathcal{X}}^\alpha(\tau)))\,d\tau\Big|\\ &\le C_T(1 + \tilde{S}(t))^2\tilde{\mathcal{E}}(t).\end{aligned}$$

Proof. Let $\alpha \in \mathcal{A}$ and let

$$z(\tau) = x - t + \tau - \mathcal{X}(\tau,0,C^\alpha)$$

and

$$\tilde{z}(\tau) = x - t + \tau - \tilde{\mathcal{X}}^\alpha(\tau).$$

Note that by (8.18)

$$z'(\tau) = 1 - v_1(\mathcal{P}(\tau, 0, C^\alpha)) \geq \frac{1}{C_T}, \tag{8.26}$$

and by (8.19)

$$\tilde{z}'(\tau) = 1 - v_1(\tilde{\mathcal{P}}^\alpha(\tau)) \geq \frac{1}{2(1 + \tilde{S}^2(\tau))}. \tag{8.27}$$

Hence for $\tau \in [0, t]$ we may define z^{-1} and $\tilde{z}^{-1}$ by

$$z^{-1}(z(\tau)) = \tau \quad \text{and} \quad \tilde{z}^{-1}(\tilde{z}(\tau)) = \tau.$$

Note that $z(0) = \tilde{z}(0)$ and write

$$\begin{aligned}
\int_0^t [\delta_\varepsilon(z(\tau)) - \delta_\varepsilon(\tilde{z}(\tau))]\, d\tau &= \int_0^t \left[\frac{\delta_\varepsilon(z(\tau))}{z'(\tau)} z'(\tau) - \frac{\delta_\varepsilon(\tilde{z}(\tau))}{\tilde{z}'(\tau)} \tilde{z}'(\tau)\right] d\tau \\
&= \int_{z(0)}^{z(t)} \frac{\delta_\varepsilon(\zeta)}{z'(z^{-1}(\zeta))}\, d\zeta - \int_{\tilde{z}(0)}^{\tilde{z}(t)} \frac{\delta_\varepsilon(\zeta)}{\tilde{z}'(\tilde{z}^{-1}(\zeta))}\, d\zeta \\
&= \int_{z(0)}^{A} \delta_\varepsilon(\zeta) \left[\frac{1}{z'(z^{-1}(\zeta))} - \frac{1}{\tilde{z}'(\tilde{z}^{-1}(\zeta))}\right] d\zeta \\
&\quad + \int_A^B \frac{\delta_\varepsilon(\zeta)}{y(\zeta)}\, d\zeta
\end{aligned} \tag{8.28}$$

where

$$A = \min\{z(t), \tilde{z}(t)\}, \qquad B = \max\{z(t), \tilde{z}(t)\},$$

and

$$y = \begin{cases} z' \circ z^{-1} & \text{if } z(t) \geq \tilde{z}(t), \\ \tilde{z}' \circ \tilde{z}^{-1} & \text{if } z(t) < \tilde{z}(t). \end{cases}$$

Next, using (8.26) and (8.27) we have for $\zeta \in [0, A]$

$$\begin{aligned}
&\left|\frac{1}{z'(z^{-1}(\zeta))} - \frac{1}{\tilde{z}'(\tilde{z}^{-1}(\zeta))}\right| = \left|\frac{z'(z^{-1}(\zeta)) - \tilde{z}'(\tilde{z}^{-1}(\zeta))}{z'(z^{-1}(\zeta))\tilde{z}'(\tilde{z}^{-1}(\zeta))}\right| \\
&\leq C_T(1 + \tilde{S}(\tau))^2 \left[|z'(z^{-1}(\zeta)) - z'(\tilde{z}^{-1}(\zeta))| + |z'(\tilde{z}^{-1}(\zeta)) - \tilde{z}'(\tilde{z}^{-1}(\zeta))|\right] \\
&\leq C_T(1 + \tilde{S}(\tau))^2 \left[2|z^{-1}(\zeta) - \tilde{z}^{-1}(\zeta)| + \sup_{[0,t]} |z' - \tilde{z}'|\right].
\end{aligned} \tag{8.29}$$

It follows from (8.26) that

$$|(z^{-1})'| \leq C_T,$$

so that

$$\begin{aligned}|z^{-1}(\zeta) - \tilde{z}^{-1}(\zeta)| &= |z^{-1}(\zeta) - z^{-1}(z(\tilde{z}^{-1}(\zeta)))| \\ &\le C_T|\zeta - z(\tilde{z}^{-1}(\zeta))| \\ &= C_T|\tilde{z}(\tilde{z}^{-1}(\zeta)) - z(\tilde{z}^{-1}(\zeta))| \\ &= C_T|\mathcal{X}(\tilde{z}^{-1}(\zeta), 0, C^\alpha) - \tilde{\mathcal{X}}^\alpha(\tilde{z}^{-1}(\zeta))| \\ &\le C_T\tilde{\mathcal{E}}(\tau).\end{aligned}$$

Also

$$\begin{aligned}|z'(\tau) - \tilde{z}'(\tau)| &= |v_1(\mathcal{P}(\tau, 0, C^\alpha)) - v_1(\tilde{\mathcal{P}}^\alpha(\tau))| \\ &\le |\mathcal{P}(\tau, 0, C^\alpha) - \tilde{\mathcal{P}}^\alpha(\tau)| \le \tilde{\mathcal{E}}(\tau),\end{aligned}$$

so (returning to (8.29)) we have

$$\begin{aligned}\left|\frac{1}{z'(z^{-1}(\zeta))} - \frac{1}{\tilde{z}'(\tilde{z}^{-1}(\zeta))}\right| &\le C_T(1 + \tilde{S}(\tau))^2[2\tilde{\mathcal{E}}(\tau) + \tilde{\mathcal{E}}(\tau)] \\ &\le C_T(1 + \tilde{S}(t))^2\tilde{\mathcal{E}}(t).\end{aligned}$$

Note also that by (8.26) and (8.27)

$$0 \le \frac{1}{y(\zeta)} \le C_T(1 + \tilde{S}(t))^2.$$

so from (8.28)

$$\begin{aligned}&\left|\int_0^t [\delta_\varepsilon(z(\tau)) - \delta_\varepsilon(\tilde{z}(\tau))]\, d\tau\right| \\ &\le \int_{z(0)}^A C_T(1 + \tilde{S}(t))^2\tilde{\mathcal{E}}(t)\delta_\varepsilon(\zeta)\, d\zeta + \int_A^B \delta_\varepsilon(\zeta)C_T(1 + \tilde{S}(t))^2\, d\zeta \\ &\le C_T(1 + \tilde{S}(t))^2 \left[\tilde{\mathcal{E}}(t) + \int_A^B \delta_\varepsilon(\zeta)\, d\zeta\right].\end{aligned} \tag{8.30}$$

Now using (8.30) for each α we have

$$\begin{aligned}&\left|\sum_\alpha q^\alpha \int_0^t [\delta_\varepsilon(x - t + \tau - \mathcal{X}(\tau, 0, C^\alpha)) - \delta_\varepsilon(x - t + \tau - \tilde{\mathcal{X}}^\alpha(\tau))]\, d\tau\right| \\ &\le C_T(1 + \tilde{S}(t))^2 \sum_\alpha q^\alpha \left[\tilde{\mathcal{E}}(t) + \int_A^B \delta_\varepsilon(\zeta)\, d\zeta\right] \\ &\le C_T(1 + \tilde{S}(t))^2 \left[\tilde{\mathcal{E}}(t) + \sum_\alpha q^\alpha \int_A^B \delta_\varepsilon(\zeta)\, d\zeta\right],\end{aligned} \tag{8.31}$$

where

$$\begin{aligned}A &= \min\{z(t), \tilde{z}(t)\} = \min\{x - \mathcal{X}(t, 0, C^\alpha), x - \tilde{\mathcal{X}}^\alpha(t)\}, \\ B &= \max\{x - \mathcal{X}(t, 0, C^\alpha), x - \tilde{\mathcal{X}}^\alpha(t)\}.\end{aligned}$$

A crucial part of the argument is an estimate of the number of nonzero terms in the sum

$$\sum_\alpha q^\alpha \int_A^B \delta_\varepsilon(\zeta)\, d\zeta.$$

We will need the following lemma.

LEMMA 8.3.2 *Given $x \in \mathbb{R}$ and $\eta > 0$ let*

$$\overline{\mathcal{A}} = \{\alpha \in \mathcal{A} : |\mathcal{X}(t, 0, C^\alpha) - x| < \eta\}.$$

Then

$$\textit{cardinality of } \overline{\mathcal{A}} \leq \frac{C_T(\eta + C_T\Delta)}{\Delta x (\Delta p)^2}.$$

Proof. By the smoothness of the solution to the original problem and by continuity with respect to initial conditions we know that

$$|\mathcal{X}(t, 0, y, p) - \mathcal{X}(t, 0, z, q)| \leq C_T(|y - z| + |p - q|). \tag{8.32}$$

Now if $\alpha \in \overline{\mathcal{A}}$ and $(z, q) \in C^\alpha$, then

$$\begin{aligned} |x - \mathcal{X}(t, 0, z, q)| &\leq |x - \mathcal{X}(t, 0, C^\alpha)| + |\mathcal{X}(t, 0, C^\alpha) - \mathcal{X}(t, 0, z, q)| \\ &< \eta + C_T\Delta. \end{aligned}$$

Hence

$$\bigcup_{\alpha \in \overline{\mathcal{A}}} \{\mathcal{X}(t, 0, z, q) : (z, q) \in C^\alpha\} \subset (x - \eta - C_T\Delta, x + \eta + C_T\Delta),$$

and

$$\begin{aligned} &\bigcup_{\alpha \in \overline{\mathcal{A}}} \{(\mathcal{X}(t, 0, z, q), \mathcal{P}(t, 0, z, q)) : (z, q) \in C^\alpha\} \\ &\qquad \subset (x - \eta - C_T\Delta, x + \eta + C_T\Delta) \times (-C_T, C_T) \times (-C_T, C_T). \end{aligned}$$

Next using the measure preserving property of $(\mathcal{X}, \mathcal{P})$, we have

$$\begin{aligned} C_T(\eta + C_T\Delta) &\geq \mu\left(\bigcup_{\alpha \in \overline{\mathcal{A}}} \{(\mathcal{X}(t, 0, z, q), \mathcal{P}(t, 0, z, q)) : (z, q) \in C^\alpha\}\right) \\ &= \mu\left(\bigcup_{\alpha \in \overline{\mathcal{A}}} C^\alpha\right) \\ &= \Delta x (\Delta p)^2 \text{card}(\overline{\mathcal{A}}), \end{aligned}$$

where "μ" denotes Lebesgue measure on $\mathbb{R}^3$. Lemma 8.3.2 now follows.

Let us return to the proof of Lemma 8.3.1. We take $\eta = \varepsilon + \tilde{\mathcal{E}}(t)$ and

$$\overline{\mathcal{A}} = \{\alpha \in \mathcal{A} : |\mathcal{X}(t,0,C^\alpha) - x| < \varepsilon + \tilde{\mathcal{E}}(t)\}.$$

Note that

$$(A,B) \subset (x - \mathcal{X}(t,0,C^\alpha) - \tilde{\mathcal{E}}(t), x - \mathcal{X}(t,0,C^\alpha) + \tilde{\mathcal{E}}(t)).$$

If $\alpha \in \mathcal{A}\backslash\overline{\mathcal{A}}$, then

$$|x - \mathcal{X}(t,0,C^\alpha)| \geq \varepsilon + \tilde{\mathcal{E}}(t)$$

and hence

$$(A,B) \cap (-\varepsilon, \varepsilon) = \emptyset.$$

But supp $\delta_\varepsilon \subset [-\varepsilon, \varepsilon]$ so if $\alpha \in \mathcal{A}\backslash\overline{\mathcal{A}}$ then

$$\int_A^B \delta_\varepsilon(\zeta)\,d\zeta = 0,$$

and hence, by Lemma 8.3.2,

$$\begin{aligned}
\sum_\alpha q^\alpha \int_A^B \delta_\varepsilon(\zeta)\,d\zeta &= \sum_{\alpha\in\overline{\mathcal{A}}} q^\alpha \int_A^B \delta_\varepsilon(\zeta)\,d\zeta \\
&\leq \operatorname{card}(\overline{\mathcal{A}}) \max_\alpha \left(q^\alpha \int_A^B \delta_\varepsilon(\zeta)\,d\zeta \right) \\
&\leq \frac{C_T(\tilde{\mathcal{E}}(t) + \varepsilon + C_T\Delta)}{\Delta x(\Delta p)^2} \max_\alpha \left(f^0(C^\alpha)\Delta x(\Delta p)^2 \int_A^B \delta_\varepsilon(\zeta)\,d\zeta \right) \\
&\leq C_T \left[\tilde{\mathcal{E}}(t) \int \delta_\varepsilon(\zeta)\,d\zeta + (\varepsilon + C_T\Delta) \max_\alpha \int_A^B \delta_\varepsilon(\zeta)\,d\zeta \right] \\
&\leq C_T[\tilde{\mathcal{E}}(t) + (\varepsilon + C_T\Delta)\|\delta_\varepsilon\|_{L^\infty} \max_\alpha (B - A)] \\
&= C_T[\tilde{\mathcal{E}}(t) + (\varepsilon + C_T\Delta)\varepsilon^{-1} \max_\alpha (B - A)] \\
&\leq C_T[\tilde{\mathcal{E}}(t) + (1 + \varepsilon^{-1}C_T\Delta)\tilde{\mathcal{E}}(t)] \\
&\leq C_T\tilde{\mathcal{E}}(t).
\end{aligned}$$

Now by (8.31) we have

$$\begin{aligned}
&\left| \sum_\alpha q^\alpha \int_0^t [\delta_\varepsilon(x - t + \tau - \mathcal{X}(\tau,0,C^\alpha)) - \delta_\varepsilon(x - t + \tau - \tilde{\mathcal{X}}^\alpha(\tau))]\,d\tau \right| \\
&\leq C_T(1 + \tilde{S}(t))^2\tilde{\mathcal{E}}(t).
\end{aligned}$$

This is the first estimate asserted in Lemma 8.3.1. The other may be established in a highly similar fashion by considering the other set of characteristics. We omit these estimates.

With Lemma 8.3.1 established we may now estimate "Sep" easily. From (8.24)

$$\begin{aligned}\mathrm{Sep} \le \sum_\alpha q^\alpha \int_0^{t^n} \delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha)) \left|v_2(\mathcal{P}(\tau,0,C^\alpha)) - v_2(\tilde{\mathcal{P}}^\alpha(\tau))\right| d\tau \\ + \left|\sum_\alpha q^\alpha \int_0^{t^n} [\delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha)) - \delta_\varepsilon(x^k - t^n + \tau - \tilde{\mathcal{X}}^\alpha(\tau))]\, d\tau\right|.\end{aligned}$$

For the first term note that for all $\alpha \in \mathcal{A}$

$$\left|v_2(\mathcal{P}(\tau,0,C^\alpha)) - v_2(\tilde{\mathcal{P}}^\alpha(\tau))\right| \le \tilde{\mathcal{E}}(\tau) \le \tilde{\mathcal{E}}(t^n)$$

and by (8.18)

$$\begin{aligned}&\int_0^{t^n} \delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha))\, d\tau \\ &\qquad \le C_T \int \delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha))(1 - v_1(\mathcal{P}(\tau,0,C^\alpha)))\, d\tau \\ &\qquad \le C_T \int \delta_\varepsilon(\zeta)\, d\zeta = C_T.\end{aligned}$$

Hence by Lemma 8.3.1

$$\begin{aligned}\mathrm{Sep} &\le \sum_\alpha q^\alpha C_T \tilde{\mathcal{E}}(t^n) + C_T(1 + \tilde{S}(t^n))^2 \tilde{\mathcal{E}}(t^n) \\ &\le C_T(1 + \tilde{S}(t^n))^2 \tilde{\mathcal{E}}(t^n).\end{aligned} \tag{8.33}$$

In order to estimate "Quad" we define (for given t^n, x^k)

$$h(x,p) = \int_0^{t^n} f^0(x,p)\delta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,x,p)) v_2(\mathcal{P}(\tau,0,x,p))\, d\tau,$$

and note that by (8.25)

$$\begin{aligned}\mathrm{Quad} &= \left|\iint h(x,p)\, dp\, dx - \sum_\alpha h(C^\alpha)\Delta x (\Delta p)^2\right| \\ &= \left|\sum_\alpha \int_{C^\alpha} (h(x,p) - h(C^\alpha))\, dp\, dx\right|.\end{aligned} \tag{8.34}$$

We may write

$$\begin{aligned}h(x,p) &= \int_0^{t^n} \frac{d}{d\tau}\left[\theta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,x,p))\right] g(\tau,x,p)\, d\tau \\ &= \theta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,x,p)) g(\tau,x,p)\Big|_{\tau=0}^{\tau=t^n} \\ &\quad - \int_0^{t^n} \theta_\varepsilon(x^k - t^n + \tau - \mathcal{X}(\tau,0,x,p))\partial_\tau g(\tau,x,p)\, d\tau,\end{aligned} \tag{8.35}$$

where

$$g(\tau,x,p)=\frac{f^0(x,p)v_2(\mathcal{P}(\tau,0,x,p))}{1-v_1(\mathcal{P}(\tau,0,x,p))}.$$

Now since $f^0\in C^2$, $v\in C^\infty$, and $\mathcal{X},\mathcal{P}\in C^2$ we have that

$$\partial_\tau\mathcal{P}=E(\tau,\mathcal{X})+B(\tau,\mathcal{X})Mv(\mathcal{P})\in C^2,$$

and hence g and $\partial_\tau g$ are C^2. Next $\theta_\varepsilon\in W^{2,\infty}$, so by (8.35), $h\in W^{2,\infty}$. Thus by Taylor's theorem and (8.34)

$$\begin{aligned}\text{Quad}&=\Big|\sum_\alpha\int_{C^\alpha}\Big[\nabla h(C^\alpha)\cdot((x,p)-C^\alpha)\\ &\qquad+\int_0^1(1-s)((x,p)-C^\alpha)^TD^2h|_{(C^\alpha+s[(x,p)-C^\alpha])}((x,p)-C^\alpha)\,ds\Big]\,dp\,dx\Big| \qquad (8.36)\\ &=\Big|\sum_\alpha\int_{C^\alpha}\int_0^1(1-s)((x,p)-C^\alpha)^TD^2h|_{(C^\alpha+s[(x,p)-C^\alpha])}((x,p)-C^\alpha)\,ds\,dp\,dx\Big|\\ &\le\sum_\alpha\int_0^1(1-s)(3\Delta)\operatorname*{ess\,sup}_{C^\alpha}|D^2h|(3\Delta)\mu(C^\alpha)\,ds\\ &=\frac{9}{2}\Delta^2(\Delta x)(\Delta p)^2\sum_\alpha\operatorname*{ess\,sup}_{C^\alpha}|D^2h|,\end{aligned}$$

where D^2h denotes the three-by-three matrix of second-order partial derivatives of h with respect to x and p.

Let ∂_1 and ∂_2 denote any of $\partial_x,\partial_{p_1},\partial_{p_2}$. Now $|\theta_\varepsilon''|=|\delta_\varepsilon'|\le 1/\varepsilon^2$ almost everywhere so

$$|\partial_1\partial_2h|\le C_T\varepsilon^{-2}\quad\text{a.e.}$$

However, at most points this bound may be improved. The most singular term of $\partial_1\partial_2h$ is

$$\begin{aligned}&\delta_\varepsilon'(x^k-t^n+\tau-\mathcal{X})(\partial_1\mathcal{X})(\partial_2\mathcal{X})g|_0^{t^n}\\ &\qquad-\int_0^{t^n}\delta_\varepsilon'(x^k-t^n+\tau-\mathcal{X})(\partial_1\mathcal{X})(\partial_2\mathcal{X})(\partial_\tau g)\,d\tau,\end{aligned} \qquad (8.37)$$

the second part of which satisfies

$$\begin{aligned}&\Big|\int_0^{t^n}\delta_\varepsilon'(x^k-t^n+\tau-\mathcal{X})(\partial_1\mathcal{X})(\partial_2\mathcal{X})(\partial_\tau g)\,d\tau\Big|\\ &\quad=\Big|\int_0^{t^n}\frac{d}{d\tau}(\delta_\varepsilon(x^k-t^n+\tau-\mathcal{X}))\frac{(\partial_1\mathcal{X})(\partial_2\mathcal{X})(\partial_\tau g)}{1-v_1(\mathcal{P})}\,d\tau\Big|\\ &\quad=\Big|\delta_\varepsilon(x^k-t^n+\tau-\mathcal{X})\frac{(\partial_1\mathcal{X})(\partial_2\mathcal{X})(\partial_\tau g)}{1-v_1(\mathcal{P})}\Big|_{\tau=0}^{\tau=t^n}\\ &\qquad-\int_0^{t^n}\delta_\varepsilon(x^k-t^n+\tau-\mathcal{X})\frac{\partial}{\partial\tau}\left(\frac{(\partial_1\mathcal{X})(\partial_2\mathcal{X})(\partial_\tau g)}{1-v_1(\mathcal{P})}\right)d\tau\Big|\\ &\quad\le C_T\varepsilon^{-1},\end{aligned}$$

where we abbreviate $\mathcal{X} = \mathcal{X}(\tau, 0, x, p)$, etc. Hence

$$\begin{aligned}(8.38)\quad \sum_{\alpha}\Big|\int_0^{t^n} \delta_\varepsilon'(x^k - t^n + \tau - \mathcal{X})(\partial_1\mathcal{X})(\partial_2\mathcal{X})(\partial_\tau g)\,d\tau\Big| &= \sum_{\alpha\in\mathcal{A}}\Big|\int_0^{t^n} \delta_\varepsilon'(x^k - t^n + \tau - \mathcal{X})(\partial_1\mathcal{X})(\partial_2\mathcal{X})(\partial_\tau g)\,d\tau\Big| \\ &\le C_T\varepsilon^{-1}\,\mathrm{card}\,(\mathcal{A}) \\ &\le C_T\varepsilon^{-1}(\Delta x)^{-1}(\Delta p)^{-2}.\end{aligned}$$

To estimate the first part of (8.37), note that for $\tau = 0$ or t^n and $(x,p)\in C^\alpha$,

$$\delta_\varepsilon'(x^k - t^n + \tau - \mathcal{X}(\tau, 0, x, p)) \ne 0$$

implies

$$|x^k - t^n + \tau - \mathcal{X}(\tau, 0, x, p)| < \varepsilon,$$

and hence by (8.32)

$$\begin{aligned}|x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha)| &\le |x^k - t^n + \tau - \mathcal{X}(\tau,0,x,p)| + |\mathcal{X}(\tau,0,x,p) - \mathcal{X}(\tau,0,C^\alpha)| \\ &< \varepsilon + C_T|(x,p) - C^\alpha| \\ &\le \varepsilon + C_T\Delta.\end{aligned}$$

Hence for $\tau \in \{0, t^n\}$

$$\begin{aligned}&\Big\{\alpha : \sup_{C^\alpha}|\delta_\varepsilon'(x^k - t^n + \tau - \mathcal{X})(\partial_1\mathcal{X})(\partial_2\mathcal{X})g| \ne 0\Big\} \\ &\quad\subset \overline{\mathcal{A}}_\tau \equiv \{\alpha\in\mathcal{A} : |x^k - t^n + \tau - \mathcal{X}(\tau,0,C^\alpha)| < \varepsilon + C_T\Delta\}\end{aligned}$$

and by Lemma 8.3.2

$$\mathrm{card}(\overline{\mathcal{A}}_\tau) \le C_T(\varepsilon + C_T\Delta)(\Delta x)^{-1}(\Delta p)^{-2}.$$

Thus (with $\overline{\mathcal{A}} = \overline{\mathcal{A}}_0 \cup \overline{\mathcal{A}}_{t^n}$) we have

$$\begin{aligned}(8.39)\quad &\sum_\alpha \operatorname*{ess\,sup}_{C^\alpha}\big|\delta_\varepsilon'(x^k - t^n + \tau - \mathcal{X})(\partial_1\mathcal{X})(\partial_2\mathcal{X})g|_0^{t^n}\big| \\ &\le (\|\delta_\varepsilon'\|_{L^\infty})C_T\,\mathrm{card}\,(\overline{\mathcal{A}}) \\ &\le \varepsilon^{-2}C_T(\varepsilon + C_T\Delta)(\Delta x)^{-1}(\Delta p)^{-2} \\ &\le C_T\varepsilon^{-1}(\Delta x)^{-1}(\Delta p)^{-2}.\end{aligned}$$

Estimates like (8.39) may be made for all the other terms of $\partial_1\partial_2 h$, so combining (8.39) and (8.38) we have

$$\sum_\alpha \operatorname*{ess\,sup}_{C^\alpha}|\partial_1\partial_2 h| \le \frac{C_T}{\varepsilon(\Delta x)(\Delta p)^2}.$$

It follows that

$$\sum_{\alpha} \operatorname*{ess\,sup}_{C^\alpha} |D^2 h| \le \frac{C_T}{\varepsilon(\Delta x)(\Delta p)^2}, \tag{8.40}$$

and hence by (8.36)

$$\text{Quad} \le C_T \varepsilon^{-1} \Delta^2. \tag{8.41}$$

Finally, we estimate "Moll". By Lemma 4.4 of [13] (with $p = +\infty$, $k = 2$, $n = 1$) we have

$$\left| j_2(\tau, z) - \int j_2(\tau, y) \delta_\varepsilon(z - y)\, dy \right| \le C\varepsilon^2 \|\partial_x^2 j_2(\tau)\|_{L^\infty}$$

for all $\tau \ge 0$ and $z \in \mathbb{R}$. Thus from (8.22) we have

$$\text{Moll} \le C\varepsilon^2 \int_0^{t^n} \|\partial_x^2 j_2(\tau)\|_{L^\infty}\, d\tau \le C_T \varepsilon^2. \tag{8.42}$$

Collecting (8.33), (8.41), and (8.42) we have by (8.21)

$$|(E_2 + B) - (\tilde{E}_2 + \tilde{B})||_{(t^n, x^k)} \le C_T \left(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + (1 + \tilde{S}(t^n))^2 \tilde{\mathcal{E}}(t^n) \right). \tag{8.43}$$

Clearly, a similar analysis of the other set of characteristics will yield

$$|(E_2 - B) - (\tilde{E}_2 - \tilde{B})||_{(t^n, x^k)} \le C_T \left(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + (1 + \tilde{S}(t^n))^2 \tilde{\mathcal{E}}(t^n) \right). \tag{8.44}$$

Now by (8.43) and (8.44)

$$\begin{aligned} |E_2 - \tilde{E}_2||_{(t^n, x^k)} &= \frac{1}{2} |[(E_2 + B) + (E_2 - B)] - [(\tilde{E}_2 + \tilde{B}) + (\tilde{E}_2 - \tilde{B})]||_{(t^n, x^k)} \\ &\le \frac{1}{2} (|(E_2 + B) - (\tilde{E}_2 + \tilde{B})||_{(t^n, x^k)} + |(E_2 - B) - (\tilde{E}_2 - \tilde{B})||_{(t^n, x^k)}) \\ &\le C_T(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + [1 + \tilde{S}(t^n)]^2 \tilde{\mathcal{E}}(t^n)), \end{aligned} \tag{8.45}$$

and similarly

$$|B - \tilde{B}||_{(t^n, x^k)} \le C_T(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + [1 + \tilde{S}(t^n)]^2 \tilde{\mathcal{E}}(t^n)). \tag{8.46}$$

Now we sketch the analogous estimate of $E_1 - \tilde{E}_1$. Let $\sigma = \int f\, dp$ so that $\rho = \sigma - \nu$. Then by (8.1) and (8.12)

$$E_1(t^n, x^k) - \tilde{E}_1(t^n, x^k) = 4\pi \int_{-\infty}^{x^k} \sigma(t^n, y)\, dy - 4\pi \sum_{\alpha} q^\alpha \theta_\varepsilon(x^k - \tilde{X}^\alpha(t^n)).$$

As before, we write

$$\frac{1}{4\pi} \left| E_1(t^n, x^k) - \tilde{E}_1(t^n, x^k) \right| \le \text{Moll}_1 + \text{Quad}_1 + \text{Sep}_1 \tag{8.47}$$

where we define

$$\begin{aligned}
\mathrm{Moll}_1 &= \left| \int_{-\infty}^{x^k} \left(\sigma(t^n, y) - \int \sigma(t^n, z)\delta_\varepsilon(y-z)\, dz \right) dy \right|, \\
\mathrm{Quad}_1 &= \left| \int_{-\infty}^{x^k} \int \sigma(t^n, z)\delta_\varepsilon(y-z)\, dz\, dy - \sum_\alpha q^\alpha \theta_\varepsilon(x^k - \mathcal{X}(t^n, 0, C^\alpha)) \right|, \\
\mathrm{Sep}_1 &= \left| \sum_\alpha q^\alpha (\theta_\varepsilon(x^k - \mathcal{X}(t^n, 0, C^\alpha)) - \theta_\varepsilon(x^k - \tilde{\mathcal{X}}^\alpha(t^n))) \right|.
\end{aligned}$$

As in the previous analysis we see that Quad_1 is equal to

$$\begin{aligned}
&\left| \int \sigma(t^n, z) \int_{-\infty}^{x^k} \delta_\varepsilon(y-z)\, dy\, dz - \sum_\alpha q^\alpha \theta_\varepsilon(x^k - \mathcal{X}(t^n, 0, C^\alpha)) \right| \\
&= \left| \iint f(t^n, z, p)\theta_\varepsilon(x^k - z)\, dp\, dz - \sum_\alpha q^\alpha \theta_\varepsilon(x^k - \mathcal{X}(t^n, 0, C^\alpha)) \right| \\
&= \left| \iint f^0(z, p)\theta_\varepsilon(x^k - \mathcal{X}(t^n, 0, z, p))\, dp\, dz - \sum_\alpha q^\alpha \theta_\varepsilon(x^k - \mathcal{X}(t^n, 0, C^\alpha)) \right|,
\end{aligned}$$

so "Quad_1" is the error in the midpoint rule applied to the function

$$(z,p) \mapsto h_1(z,p) \equiv f^0(z,p)\theta_\varepsilon(x^k - \mathcal{X}(t^n, 0, z, p)). \tag{8.48}$$

In order to estimate "Moll_1" note first that

$$\begin{aligned}
\int_{-\infty}^{x^k} \int \sigma(t^n, z)\delta_\varepsilon(y-z)\, dz\, dy &= \iint_{-\infty}^{x^k} \sigma(t^n, z)\delta_\varepsilon(y-z)\, dy\, dz \\
&= \iint_z^{\infty} \sigma(t^n, z)\delta_\varepsilon(x^k - \nu)\, d\nu\, dz \\
&= \iint_{-\infty}^{\nu} \sigma(t^n, z)\delta_\varepsilon(x^k - \nu)\, dz\, d\nu,
\end{aligned}$$

where we have used the substitution

$$\nu = x^k + z - y.$$

So by its definition

$$\mathrm{Moll}_1 = \left| \int_{-\infty}^{x^k} \sigma(t^n, z)\, dz - \iint_{-\infty}^{\nu} \sigma(t^n, z)\, dz \delta_\varepsilon(x^k - \nu)\, d\nu \right|.$$

Now we again use Lemma 4.4 of [13] and obtain

$$\mathrm{Moll}_1 \leq C\varepsilon^2 \left\| \partial_\nu^2 \left(\int_{-\infty}^{\nu} \sigma(t^n, z)\, dz \right) \right\|_{L^\infty} \leq C_T \varepsilon^2. \tag{8.49}$$

For "Quad$_1$" we again have (as in (8.36))

$$\begin{aligned}\mathrm{Quad}_1 &= \Big|\iint h_1\, dp\, dz - \sum_\alpha h_1(C^\alpha)(\Delta p)^2 \Delta x\Big| \\ &\le \frac{9}{2}\Delta^2(\Delta x)(\Delta p)^2 \sum_\alpha \operatorname*{ess\,sup}_{C^\alpha} |D^2 h_1|.\end{aligned}$$

Again $|D^2 h_1| \le C_T \varepsilon^{-2}$ holds almost everywhere, but an improved bound is possible at most points. Let

$$\overline{\mathcal{A}} := \{\alpha \in \mathcal{A} : \text{ there exists } \quad (y,p) \in C^\alpha \quad \text{such that} \quad |\mathcal{X}(t^n, 0, y, p) - x^k| < \varepsilon\}.$$

By Lemma 8.3.2 we have

$$\operatorname{card}(\overline{\mathcal{A}}) \le C_T(\varepsilon + C_T\Delta)(\Delta x)^{-1}(\Delta p)^{-2}.$$

Moreover we have from (8.40)

$$\sum_\alpha \operatorname*{ess\,sup}_{C^\alpha} |D^2 h_1| \le \frac{C_T}{\varepsilon(\Delta x)(\Delta p)^2}. \tag{8.50}$$

Hence

$$\begin{aligned}\mathrm{Quad}_1 &\le \frac{9}{2}\Delta^2(\Delta x)(\Delta p)^2 \sum_\alpha |D^2 h_1| \\ &\le \frac{9}{2}\Delta^2(\Delta x)(\Delta p)^2 \left[\sum_{\alpha\in\overline{\mathcal{A}}} + \sum_{\alpha\in\mathcal{A}-\overline{\mathcal{A}}}\right] |D^2 h_1| \\ &\le \frac{9}{2}\Delta^2(\Delta x)(\Delta p)^2 \left(C_T\varepsilon^{-2}\operatorname{card}(\overline{\mathcal{A}}) + C_T\varepsilon^{-1}C(\Delta x)^{-1}(\Delta p)^{-2}\right) \qquad (8.51) \\ &\le C_T\Delta^2(\varepsilon^{-2}(\varepsilon + C_T\Delta) + \varepsilon^{-1}) \\ &\le C_T\Delta^2\varepsilon^{-1}.\end{aligned}$$

In order to estimate "Sep$_1$", let

$$\overline{\mathcal{A}} = \{\alpha \in \mathcal{A} : |x^k - \mathcal{X}(t^n, 0, C^\alpha)| < \varepsilon + \tilde{\mathcal{E}}(t^n)\}$$

and note that if $\alpha \in \mathcal{A}\backslash\overline{\mathcal{A}}$ then either

$$(\mathcal{X}(t^n, 0, C^\alpha) > x^k + \varepsilon \quad \text{and} \quad \tilde{\mathcal{X}}^\alpha(t^n) > x^k + \varepsilon)$$

or

$$(\mathcal{X}(t^n, 0, C^\alpha) < x^k - \varepsilon \quad \text{and} \quad \tilde{\mathcal{X}}^\alpha(t^n) < x^k - \varepsilon).$$

In either case

$$\theta_\varepsilon(x^k - \mathcal{X}(t^n, 0, C^\alpha)) - \theta_\varepsilon(x^k - \tilde{\mathcal{X}}^\alpha(t^n)) = 0.$$

Now by Lemma 8.3.2

$$\operatorname{card}(\overline{\mathcal{A}}) \le C_T(\varepsilon + C_T\tilde{\mathcal{E}}(t^n))(\Delta x)^{-1}(\Delta p)^{-2},$$

and by definition

$$\begin{aligned}\mathrm{Sep}_1 &= \Big| \sum_{\alpha\in\overline{\mathcal{A}}} q^\alpha(\theta_\varepsilon(x^k - \mathcal{X}(t^n,0,C^\alpha)) - \theta_\varepsilon(x^k - \tilde{\mathcal{X}}^\alpha(t^n)))\Big| \\ &\le \sum_{\alpha\in\overline{\mathcal{A}}} q^\alpha \int_A^B \delta_\varepsilon(\zeta)\,d\zeta,\end{aligned}$$

where

$$\begin{aligned}A &= \min\{x^k - \mathcal{X}(t^n,0,C^\alpha),\ x^k - \tilde{\mathcal{X}}^\alpha(t^n)\},\\ B &= \max\{x^k - \mathcal{X}(t^n,0,C^\alpha),\ x^k - \tilde{\mathcal{X}}^\alpha(t^n)\}.\end{aligned}$$

Hence

$$\begin{aligned}\mathrm{Sep}_1 &\le \operatorname{card}(\overline{\mathcal{A}}) \max_\alpha \left(q^\alpha \int_A^B \delta_\varepsilon(\zeta)\,d\zeta \right) \\ &\le C_T(\varepsilon + C_T\tilde{\mathcal{E}}(t^n)) \max_\alpha \left(\int_A^B \delta_\varepsilon(\zeta)\,d\zeta \right) \\ &\le C_T \left(\varepsilon\|\delta_\varepsilon\|_{L^\infty} \max_\alpha(B-A) + C_T\tilde{\mathcal{E}}(t^n) \int \delta_\varepsilon(\zeta)\,d\zeta \right) \\ &= C_T \left(\max_\alpha(B-A) + C_T\tilde{\mathcal{E}}(t^n) \right) \\ &\le C_T\tilde{\mathcal{E}}(t^n).\end{aligned} \tag{8.52}$$

Collecting (8.49) – (8.52) we have

$$|E_1(t^n,x^k) - \tilde{E}_1(t^n,x^k)| \le C_T(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + \tilde{\mathcal{E}}(t^n)). \tag{8.53}$$

We will need estimates of the field errors at all values of x (not just grid-points), so we use the following lemma.

LEMMA 8.3.3 *Let $g \in C^2(\mathbb{R})$. For any real values a, b, G_a, G_b with $a < b$ we have*

$$\begin{aligned}\left| g(x) - \left[G_a\frac{b-x}{b-a} + G_b\frac{x-a}{b-a} \right] \right| &\le \frac{1}{8} \max_{\xi\in[a,b]} |g''(\xi)|(b-a)^2 \\ &\quad + \frac{b-x}{b-a}|G_a - g(a)| + \frac{x-a}{b-a}|G_b - g(b)|\end{aligned}$$

for all $x \in [a,b]$.

Proof. Let $x \in [a,b]$; then there exists $\xi \in [a,b]$ such that

$$\begin{aligned}
&\left|g(x) - \left[G_a\frac{b-x}{b-a} + G_b\frac{x-a}{b-a}\right]\right| \\
&\le \left|g(x) - \left[g(a)\frac{b-x}{b-a} + g(b)\frac{x-a}{b-a}\right]\right| \\
&\quad + \left|\left[g(a)\frac{b-x}{b-a} + g(b)\frac{x-a}{b-a}\right] - \left[G_a\frac{b-x}{b-a} + G_b\frac{x-a}{b-a}\right]\right| \\
&= \frac{1}{2}|g''(\xi)|(b-x)(x-a) + \left|\frac{b-x}{b-a}(g(a)-G_a) + \frac{x-a}{b-a}(g(b)-G_b)\right| \\
&\le \frac{1}{8}|g''(\xi)|(b-a)^2 + \frac{b-x}{b-a}|g(a)-G_a| + \frac{x-a}{b-a}|g(b)-G_b|,
\end{aligned}$$

which establishes the lemma.

Recall that E and B are defined for $x \in (x^k, x^{k+1})$ by linear interpolation. Let $b = x^{k+1}$ and $a = x^k$; then use of (8.45) and Lemma 8.3.3 yield

$$\begin{aligned}
|E_2 - \tilde{E}_2||_{(t^n,x)} &\le \frac{1}{8}\|\partial_x^2 E_2(t^n)\|_{L^\infty}\varepsilon^2 + \frac{x^{k+1}-x}{\varepsilon}|E_2 - \tilde{E}_2||_{(t^n,x^k)} \\
&\quad + \frac{x-x^k}{\varepsilon}|E_2 - \tilde{E}_2||_{(t^n,x^{k+1})} \\
&\le C_T\varepsilon^2 + \left(\frac{x^{k+1}-x}{\varepsilon} + \frac{x-x^k}{\varepsilon}\right)\left(C_T[\varepsilon^2 + \varepsilon^{-1}\Delta^2 + (1+\tilde{S}(t^n))^2\tilde{\mathcal{E}}(t^n)]\right) \\
&\le C_T(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + (1+\tilde{S}(t^n))^2\tilde{\mathcal{E}}(t^n)).
\end{aligned}$$

Similarly, using (8.45), (8.46) and (8.53) we have the following corollary.

COROLLARY *For all* $x \in \mathbb{R}$

$$\left(|E_2 - \tilde{E}_2| + |B - \tilde{B}| + |E_1 - \tilde{E}_1|\right)|_{(t^n,x)} \le C_T(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + (1+\tilde{S}(t^n))^2\tilde{\mathcal{E}}(t^n)).$$

8.4. The Particle Errors

In this section we estimate

$$|\mathcal{X}(t,0,C^\alpha) - \tilde{\mathcal{X}}^\alpha(t)| + |\mathcal{P}(t,0,C^\alpha) - \tilde{\mathcal{P}}^\alpha(t)|.$$

We start by estimating

$$|\mathcal{X}(t^{n+1},0,C^\alpha) - \tilde{\mathcal{X}}^\alpha(t^{n+1})|.$$

Toward this goal we define the truncation error to be

$$\mathcal{T}_x = \mathcal{X}(t^{n+1},0,C^\alpha) - \mathcal{X}(t^n,0,C^\alpha) - \varepsilon v_1(\mathcal{P}(t^{n+1/2},0,C^\alpha)),$$

and note that since $\mathcal{T}_x$ involves only the exact solution (which is smooth), we have

$$|\mathcal{T}_x| \le C_T\varepsilon^3.$$

Now by (8.10)

(8.54)
$$
\begin{aligned}
&|\mathcal{X}(t^{n+1},0,C^\alpha)-\tilde{\mathcal{X}}^\alpha(t^{n+1})|\\
&=\left|[\mathcal{X}(t^n,0,C^\alpha)+\varepsilon v_1(\mathcal{P}(t^{n+1/2},0,C^\alpha))+\mathcal{T}_x]-[\tilde{\mathcal{X}}^\alpha(t^n)+\varepsilon v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))]\right|\\
&\le|\mathcal{X}(t^n,0,C^\alpha)-\tilde{\mathcal{X}}^\alpha(t^n)|+\varepsilon|v_1(\mathcal{P}(t^{n+1/2},0,C^\alpha))-v_1(\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))|+|\mathcal{T}_x|\\
&\le|\mathcal{X}(t^n,0,C^\alpha)-\tilde{\mathcal{X}}^\alpha(t^n)|+\varepsilon|\mathcal{P}(t^{n+1/2},0,C^\alpha)-\tilde{\mathcal{P}}^\alpha(t^{n+1/2})|+C_T\varepsilon^3.
\end{aligned}
$$

Next we obtain an estimate analogous to (8.54) for

$$|\mathcal{P}(t^{n+3/2},0,C^\alpha)-\tilde{\mathcal{P}}^\alpha(t^{n+3/2})|.$$

Define $Q:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ and $\tilde{Q}:\mathbb{R}^2\times\mathbb{R}^2\to\mathbb{R}^2$ by

$$Q(x,p)=p+\varepsilon E(t^{n+1},x)+\varepsilon B(t^{n+1},x)M\frac{1}{2}\left(v(p)+v(Q(x,p))\right) \tag{8.55}$$

and

$$\tilde{Q}(x,p)=p+\varepsilon\tilde{E}(t^{n+1},x)+\varepsilon\tilde{B}(t^{n+1},x)M\frac{1}{2}\left(v(p)+v(\tilde{Q}(x,p))\right), \tag{8.56}$$

so that the definition (8.16) of $\tilde{\mathcal{P}}^\alpha(t^{n+3/2})$ becomes

$$\tilde{\mathcal{P}}^\alpha(t^{n+3/2})=\tilde{Q}(\tilde{\mathcal{X}}^\alpha(t^{n+1}),\tilde{\mathcal{P}}^\alpha(t^{n+1/2})).$$

Q and $\tilde{Q}$ may be shown to be well defined by explicitly solving (8.55) and (8.56), as described following (8.16). We also define the truncation error to be

$$\mathcal{T}_p=\mathcal{P}(t^{n+3/2},0,C^\alpha)-Q(\mathcal{X}(t^{n+1},0,C^\alpha),\mathcal{P}(t^{n+1/2},0,C^\alpha)), \tag{8.57}$$

so the error equation is

$$
\begin{aligned}
\mathcal{P}(t^{n+3/2},0,C^\alpha)-\tilde{\mathcal{P}}^\alpha(t^{n+3/2})&=Q(\mathcal{X}(t^{n+1},0,C^\alpha),\mathcal{P}(t^{n+1/2},0,C^\alpha))\\
&\quad-\tilde{Q}(\tilde{\mathcal{X}}^\alpha(t^{n+1}),\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))+\mathcal{T}_p.
\end{aligned} \tag{8.58}
$$

We will estimate the error by breaking up (8.58) as follows:

$$
\begin{aligned}
&|\mathcal{P}(t^{n+3/2},0,C^\alpha)-\tilde{\mathcal{P}}^\alpha(t^{n+3/2})|\\
&\le\left|Q(\mathcal{X}(t^{n+1},0,C^\alpha),\mathcal{P}(t^{n+1/2},0,C^\alpha))-Q(\tilde{\mathcal{X}}^\alpha(t^{n+1}),\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))\right|\\
&\quad+\left|Q(\tilde{\mathcal{X}}^\alpha(t^{n+1}),\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))-\tilde{Q}(\tilde{\mathcal{X}}^\alpha(t^{n+1}),\tilde{\mathcal{P}}^\alpha(t^{n+1/2}))\right|+|\mathcal{T}_p|.
\end{aligned} \tag{8.59}
$$

LEMMA 8.4.1 *For all $x,y\in\mathbb{R}$ and $p,w\in\mathbb{R}^2$*

$$|Q(x,p)-Q(y,p)|\le C_T\varepsilon|x-y|$$

and

$$|Q(x,p) - Q(x,w)| \leq (1 + 2|B(t^{n+1},x)|\varepsilon)|p - w|.$$

Proof. Differentiating (8.55) with respect to x we get

$$\partial_x Q = \varepsilon\left(\partial_x E + (\partial_x B)M\frac{1}{2}\left[v(p) + v(Q)\right] + \frac{1}{2}BM(Dv)\partial_x Q\right),$$

where we abbreviate $Q = Q(x,p)$, $E = E(t^{n+1},x)$, etc., and Dv denotes a two-by-two matrix. Hence

$$\begin{aligned}|\partial_x Q| &\leq \varepsilon\left(|\partial_x E| + |\partial_x B| + \frac{1}{2}|B||\partial_x Q|\right)\\ &\leq C_T\varepsilon + \frac{1}{2}|B|\varepsilon|\partial_x Q|.\end{aligned}$$

Recall that

$$|B|\varepsilon < 1$$

was assumed in the statement of Theorem 8.2.1, so

$$\frac{1}{2}|\partial_x Q| \leq \left(1 - \frac{1}{2}|B|\varepsilon\right)|\partial_x Q| \leq C_T\varepsilon,$$

and hence

$$|Q(x,p) - Q(y,p)| \leq C_T\varepsilon|x - y|.$$

Next by (8.55) we have

$$\begin{aligned}|Q(x,p) - Q(x,w)| &= \Bigg|\left(p + \varepsilon E + \varepsilon BM\frac{1}{2}\left[v(p) + v(Q(x,p))\right]\right)\\ &\qquad - \left(w + \varepsilon E + \varepsilon BM\frac{1}{2}\left[v(w) + v(Q(x,w))\right]\right)\Bigg|\\ &\leq |p - w| + \varepsilon|B|\frac{1}{2}\left(|v(p) - v(w)| + |v(Q(x,p)) - v(Q(x,w))|\right)\\ &\leq |p - w| + \frac{1}{2}|B|\varepsilon\left(|p - w| + |Q(x,p) - Q(x,w)|\right).\end{aligned}$$

Again using the assumption that $\varepsilon|B| < 1$, we have

$$\begin{aligned}|Q(x,p) - Q(x,w)| &\leq \frac{\left(1 + \frac{1}{2}|B|\varepsilon\right)|p - w|}{1 - \frac{1}{2}|B|\varepsilon}\\ &\leq (1 + 2|B|\varepsilon)|p - w|,\end{aligned}$$

which completes the proof of Lemma 8.4.1.

LEMMA 8.4.2 *For all* $(x,p) \in \mathbb{R}^3$

$$|Q(x,p) - \tilde{Q}(x,p)| \leq 2\varepsilon(|E - \tilde{E}| + |B - \tilde{B}|)|_{(t^{n+1},x)}.$$

Proof. By (8.55) and (8.56)

$$
\begin{aligned}
|Q(x,p)-\tilde{Q}(x,p)| &= \left|[p+\varepsilon E+\varepsilon BM\frac{1}{2}(v(p)+v(Q))]-[p+\varepsilon\tilde{E}+\varepsilon\tilde{B}M\frac{1}{2}(v(p)+v(\tilde{Q}))]\right| \\
&\le \varepsilon|E-\tilde{E}|+\frac{1}{2}\varepsilon\left[|B-\tilde{B}||v(p)|+|B||v(Q)-v(\tilde{Q})|+|B-\tilde{B}||v(\tilde{Q})|\right] \\
&\le \varepsilon(|E-\tilde{E}|+|B-\tilde{B}|)+\frac{1}{2}\varepsilon|B||Q-\tilde{Q}|.
\end{aligned}
$$

Again since $\varepsilon|B|<1$ we have

$$
\begin{aligned}
\frac{1}{2}|Q-\tilde{Q}| &\le \left(1-\frac{1}{2}\varepsilon|B|\right)|Q-\tilde{Q}| \\
&\le \varepsilon(|E-\tilde{E}|+|B-\tilde{B}|),
\end{aligned}
$$

and the lemma follows.

LEMMA 8.4.3 *For all $\alpha\in\mathcal{A}$*

$$|\mathcal{T}_p|\le C_T\varepsilon^3,$$

where $\mathcal{T}_p$ is defined in (8.57).

Proof. Let $\alpha\in\mathcal{A}$. It follows from Taylor's theorem that

$$
\begin{aligned}
&\left|\mathcal{P}(t^{n+3/2},0,C^\alpha)-\mathcal{P}(t^{n+1/2},0,C^\alpha)-\varepsilon\frac{d\mathcal{P}}{ds}(t^{n+1},0,C^\alpha)\right| \\
&\le \frac{1}{3}\varepsilon^3\max\left\{\left|\frac{d^3\mathcal{P}}{ds^3}(\tau,0,C^\alpha)\right|:\tau\in[t^{n+1/2},t^{n+3/2}]\right\} \\
&\le C_T\varepsilon^3.
\end{aligned}
\tag{8.60}
$$

We put

$$\mathcal{V}(t)=v(\mathcal{P}(t,0,C^\alpha));$$

then we also have

$$
\begin{aligned}
&\left|\frac{1}{2}[\mathcal{V}(t^{n+1/2})+\mathcal{V}(t^{n+3/2})]-\mathcal{V}(t^{n+1})\right| \\
&\le \frac{1}{2}\varepsilon^2\max\left\{\left|\frac{d^2\mathcal{V}}{d\tau^2}(\tau)\right|:\tau\in[t^{n+1/2},t^{n+3/2}]\right\} \\
&\le C_T\varepsilon^2.
\end{aligned}
\tag{8.61}
$$

Now since

$$\frac{d\mathcal{P}}{ds}(t^{n+1},0,C^\alpha)=E+BMv(\mathcal{P}(t^{n+1},0,C^\alpha))$$

(where $E=E(t^{n+1},\mathcal{X}(t^{n+1},0,C^\alpha))$ etc.), (8.60) and (8.61) yield

$$
\begin{aligned}
&\left|\mathcal{P}(t^{n+3/2},0,C^\alpha)-\left[\mathcal{P}(t^{n+1/2},0,C^\alpha)+\varepsilon\left(E+BM\frac{1}{2}[\mathcal{V}(t^{n+1/2})+\mathcal{V}(t^{n+3/2})]\right)\right]\right| \\
&\le C_T\varepsilon^3+\left|\varepsilon BM\left(\frac{1}{2}[\mathcal{V}(t^{n+1/2})+\mathcal{V}(t^{n+3/2})]-\mathcal{V}(t^{n+1})\right)\right| \\
&\le C_T\varepsilon^3.
\end{aligned}
\tag{8.62}
$$

Let us abbreviate

$$Q = Q(\mathcal{X}(t^{n+1}, 0, C^\alpha), \mathcal{P}(t^{n+1/2}, 0, C^\alpha)),$$

so that by (8.57)

$$\mathcal{T}_p = \mathcal{P}(t^{n+3/2}, 0, C^\alpha) - Q.$$

Now by (8.62) and (8.55)

$$\begin{aligned}
|\mathcal{T}_p| &\le C_T\varepsilon^3 + \left|\mathcal{P}(t^{n+1/2}, 0, C^\alpha) + \varepsilon\left(E + \frac{1}{2}BM[\mathcal{V}(t^{n+1/2}) + \mathcal{V}(t^{n+3/2})]\right) - Q\right| \\
&= C_T\varepsilon^3 + \frac{1}{2}\varepsilon\left|BM(\mathcal{V}(t^{n+3/2}) - v(Q))\right| \\
&= C_T\varepsilon^3 + \frac{1}{2}\varepsilon|B|\left|v(\mathcal{P}(t^{n+3/2}, 0, C^\alpha)) - v(Q)\right| \\
&\le C_T\varepsilon^3 + \frac{1}{2}\varepsilon|B|\left|\mathcal{P}(t^{n+3/2}, 0, C^\alpha) - Q\right| \\
&= C_T\varepsilon^3 + \frac{1}{2}\varepsilon|B||\mathcal{T}_p|.
\end{aligned}$$

Again since $\varepsilon|B| < 1$ we have

$$\frac{1}{2}|\mathcal{T}_p| \le \left(1 - \frac{1}{2}\varepsilon|B|\right)|\mathcal{T}_p| \le C_T\varepsilon^3,$$

from which the lemma follows.

Returning to (8.59), we use Lemmas 8.4.1–8.4.3 and obtain

$$\begin{aligned}
&\left|\mathcal{P}(t^{n+3/2}, 0, C^\alpha) - \tilde{\mathcal{P}}^\alpha(t^{n+3/2})\right| \\
&\le C_T\varepsilon\left|\mathcal{X}(t^{n+1}, 0, C^\alpha) - \tilde{\mathcal{X}}^\alpha(t^{n+1})\right| \\
&\qquad + (1 + 2|B(t^{n+1}, \tilde{\mathcal{X}}^\alpha(t^{n+1}))|\varepsilon)\left|\mathcal{P}(t^{n+1/2}, 0, C^\alpha) - \tilde{\mathcal{P}}^\alpha(t^{n+1/2})\right| \\
&\qquad + 2\varepsilon\left(\|(E - \tilde{E})(t^{n+1})\|_{L^\infty} + \|(B - \tilde{B})(t^{n+1})\|_{L^\infty}\right) + C_T\varepsilon^3.
\end{aligned}$$

Using $|B(t, x)| \le C$ for all t, x and the corollary to Lemma 8.3.3, we have

$$\begin{aligned}
|\mathcal{P}(t^{n+3/2}, 0, C^\alpha) - \tilde{\mathcal{P}}^\alpha(t^{n+3/2})| &\le C_T\varepsilon|\mathcal{X}(t^{n+1}, 0, C^\alpha) - \tilde{\mathcal{X}}^\alpha(t^{n+1})| \\
&\quad + (1 + C\varepsilon)|\mathcal{P}(t^{n+1/2}, 0, C^\alpha) - \tilde{\mathcal{P}}^\alpha(t^{n+1/2})| \\
&\quad + C_T\varepsilon(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + [1 + \tilde{S}(t^{n+1})]^2\tilde{\mathcal{E}}(t^{n+1})).
\end{aligned} \tag{8.63}$$

8.5. Summing the Errors

Estimates (8.54) and (8.63) bound the errors at $t = t^{n+1}$ and $t = t^{n+3/2}$ in terms of errors that occurred earlier. Unfortunately, the bound on the field errors required estimates of errors at *all* times (see (8.20) and Lemma 8.3.1), not just at $t^1, t^{3/2}, t^2, t^{5/2}$, etc. Thus we define

$$\tilde{\mathcal{E}}^m = \max\Big\{ \max\{|\mathcal{X}(\tau,0,C^\alpha) - \tilde{\mathcal{X}}^\alpha(\tau)| : \alpha \in \mathcal{A},\ \tau \in \{t^0, t^1, t^2, \ldots\} \cap [0,t^m]\},$$
$$\max\{|\mathcal{P}(\tau,0,C^\alpha) - \tilde{\mathcal{P}}^\alpha(\tau)| : \alpha \in \mathcal{A}, \tau \in \{t^{1/2}, t^{3/2}, t^{5/2}, \ldots\} \cap [0,t^m]\}\Big\}$$

for $m = 0, \frac{1}{2}, 1, \frac{3}{2}, \ldots$. Note that

$$\tilde{\mathcal{E}}^0 = \tilde{\mathcal{E}}^{1/2} = 0.$$

Now for every $\alpha \in \mathcal{A}$ and $m \in \{1, \frac{3}{2}, 2, \ldots\}$,

$$|\mathcal{X}(t,0,C^\alpha) - \mathcal{X}(t^m,0,C^\alpha)| + |\mathcal{P}(t,0,C^\alpha) - \mathcal{P}(t^m,0,C^\alpha)| \le C_T\varepsilon$$

for each $t \in [t^{m-1}, t^m]$, so that

$$\tilde{\mathcal{E}}(t^m) \le \tilde{\mathcal{E}}^m + C_T\varepsilon. \tag{8.64}$$

By (8.54) we have

$$|\mathcal{X}(t^{n+1},0,C^\alpha) - \tilde{\mathcal{X}}^\alpha(t^{n+1})| \le (1+\varepsilon)\tilde{\mathcal{E}}^{n+1/2} + C_T\varepsilon^3,$$

and using (8.64) in (8.63) we have

$$\begin{aligned}
&|\mathcal{P}(t^{n+3/2},0,C^\alpha) - \tilde{\mathcal{P}}^\alpha(t^{n+3/2})| \\
&\le C_T\varepsilon\tilde{\mathcal{E}}^{n+1} + (1+C\varepsilon)\tilde{\mathcal{E}}^{n+1/2} \\
&\quad + C_T\varepsilon\Big(\varepsilon^2 + \varepsilon^{-1}\Delta^2 + [1+\tilde{S}(t^{n+1})]^2[\tilde{\mathcal{E}}^{n+1} + C_T\varepsilon]\Big) \\
&\le \Big(1 + C_T[1+\tilde{S}(t^{n+1})]^2\varepsilon\Big)\tilde{\mathcal{E}}^{n+1} + C_T([1+\tilde{S}(t^{n+1})]^2\varepsilon^2 + \Delta^2)
\end{aligned}$$

for all $\alpha \in \mathcal{A}$ and $n \in \{0,1,2,3,\ldots\}$. It follows from the definition of $\tilde{\mathcal{E}}^m$ that

$$\tilde{\mathcal{E}}^{m+1/2} \le (1 + C_T[1+\tilde{S}(t^m)]^2\varepsilon)\tilde{\mathcal{E}}^m + C_T([1+\tilde{S}(t^m)]^2\varepsilon^2 + \Delta^2) \tag{8.65}$$

for all $m \in \{0, \frac{1}{2}, 1, \frac{3}{2}, \ldots\}$. Let $m \in \{\frac{1}{2}, 1, \frac{3}{2}, 2, \ldots\}$ be given and let

$$A_1 = \max\left\{ C_T[1+\tilde{S}(t^k)]^2 : k \in \left\{0, \frac{1}{2}, 1, \ldots, m\right\}\right\}$$

and

$$A_2 = \max\left\{ C_T([1+\tilde{S}(t^k)]^2\varepsilon^2 + \Delta^2) : k \in \left\{0, \frac{1}{2}, 1, \ldots, m\right\}\right\}.$$

Then by (8.65)

$$\begin{aligned}
\tilde{\mathcal{E}}^m &\le (1 + A_1\varepsilon)\tilde{\mathcal{E}}^{m-1/2} + A_2 \\
&\le (1 + A_1\varepsilon)^2\tilde{\mathcal{E}}^{m-1} + [(1 + A_1\varepsilon) + 1]A_2 \\
&\le \cdots \\
&\le (1 + A_1\varepsilon)^{2m}\tilde{\mathcal{E}}^0 + [(1 + A_1\varepsilon)^{2m-1} + \cdots + 1]A_2 \\
&= \left(\frac{(1 + A_1\varepsilon)^{2m} - 1}{A_1\varepsilon}\right) A_2 \\
&\le \left(1 + \frac{2mA_1\varepsilon}{2m}\right)^{2m} A_2\varepsilon^{-1}A_1^{-1} \\
&\le \exp(2mA_1\varepsilon)A_2\varepsilon^{-1}A_1^{-1} \\
&= \exp(2A_1t^m)A_2\varepsilon^{-1}A_1^{-1}.
\end{aligned}$$

Substituting the above values for A_1 and A_2 we get

$$\tilde{\mathcal{E}}^m \le \exp\left(C_T[1 + \sup_{k\le m} \tilde{S}(t^k)]^2\right)(\varepsilon + \varepsilon^{-1}\Delta^2).$$

Hence by (8.64)

$$\tilde{\mathcal{E}}(t^m) \le \exp\left(C_T[1 + \sup_{k\le m} \tilde{S}(t^k)]^2\right)(\varepsilon + \varepsilon^{-1}\Delta^2),$$

and since $\tilde{\mathcal{E}}$ is nondecreasing we have

$$\tilde{\mathcal{E}}(t) \le \exp\left(C_T[1 + \sup_{\tau\le t} \tilde{S}(\tau)]^2\right)(\varepsilon + \varepsilon^{-1}\Delta^2). \tag{8.66}$$

We comment that if $\tilde{\mathcal{E}}^0$ and $\tilde{\mathcal{E}}^{1/2}$ were not zero, then (proceeding as above) (8.66) would be

$$\tilde{\mathcal{E}}(t) \le \exp\left(C_T[1 + \sup_{\tau\le t} \tilde{S}(\tau)]^2\right)(\varepsilon + \varepsilon^{-1}\Delta^2 + \tilde{\mathcal{E}}^0 + \tilde{\mathcal{E}}^{1/2}).$$

Thus (8.66) still holds if

$$\tilde{\mathcal{E}}^0 + \tilde{\mathcal{E}}^{1/2} \le C(\varepsilon + \varepsilon^{-1}\Delta^2).$$

Next we use (8.66) to estimate $\tilde{S}(t)$. Note that for all $\alpha \in \mathcal{A}$ and $\tau \le t$

$$\begin{aligned}
|\tilde{\mathcal{P}}^\alpha(\tau)| &\le |\tilde{\mathcal{P}}^\alpha(\tau) - \mathcal{P}(\tau, 0, C^\alpha)| + |\mathcal{P}(\tau, 0, C^\alpha)| \\
&\le \tilde{\mathcal{E}}(\tau) + S(\tau) \\
&\le C_T + \tilde{\mathcal{E}}(t),
\end{aligned}$$

so that

$$\sup_{\tau<t} \tilde{S}(\tau) \le C_T + \tilde{\mathcal{E}}(t).$$

Define

$$T_{\varepsilon,\Delta} = \sup\{t : \tilde{\mathcal{E}}(\tau) \leq 1 \quad \text{for all} \quad \tau \in [0,t]\}.$$

Then

$$\sup_{\tau \leq t} \tilde{S}(\tau) \leq C_T \tag{8.67}$$

for $t \leq T \leq T_{\varepsilon,\Delta}$. Hence

$$\tilde{\mathcal{E}}(t) \leq C_T(\varepsilon + \varepsilon^{-1}\Delta^2) \tag{8.68}$$

for $t \leq T \leq T_{\varepsilon,\Delta}$. But since

$$\lim_{\substack{\varepsilon \to 0^+ \\ \Delta < C_1 \varepsilon}} C_T(\varepsilon + \varepsilon^{-1}\Delta^2) = 0,$$

it follows from (8.68) and the definition of $T_{\varepsilon,\Delta}$ that

$$\lim_{\substack{\varepsilon \to 0^+ \\ \Delta < C_1 \varepsilon}} T_{\varepsilon,\Delta} = +\infty.$$

Finally the validity of the main theorem 8.2.1 follows from (8.67), (8.68) and the corollary to Lemma 8.3.3.

We conclude with two comments regarding this scheme and related finite difference schemes. We note that Theorem 8.2.1 provides no approximations for the density ρ. To this date we have been unsuccessful in attempts to derive such estimates. Secondly, in [1] finite differences are used to advance the fields in time. This is natural since the fields satisfy linear ordinary differential equations along the characteristics. However, we have been unable to prove that such schemes converge. The scheme employed in this chapter uses the exact solution representation, applied to approximate sources. This introduces one extra order of smoothing, and allows us to close the loop of estimates.

For another approximation (which is related to the Darwin model), see [10] and [11].

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